Vol. 58, No. 1 DUKE MATHEMATICAL JOURNAL © February 1989

NONLOCAL INVERSION FORMULAS FOR THE X-RAY TRANSFORM

ALLAN GREENLEAF AND GUNTHER UHLMANN

§1. Introduction. Consider the problem of reconstructing a function $f(x)$ on $\mathbb{R}^n$ from knowledge of its integrals over k-dimensional planes. Let $M_{k,n}$ be the $(k+1)(n-k)$-dimensional bundle of (affine) k-planes in $\mathbb{R}^n$, and define the k-plane transform

$$\mathscr{R}_{k,n}f(\pi) = \int_\pi f(x)\,d\mu_\pi(x),\ \pi \in M_{k,n}, \tag{1.1}$$

where $d\mu_\pi(x)$ is Lebesgue measure on π. When $k = n - 1$, $\mathscr{R}_{k,n}$ is the classical Radon transform and $\mathscr{R}_{n-1,n}$ is an isomorphism on the appropriate spaces of smooth functions of compact support which extends to a unitary operator [He]. When $1 \leqslant k < n-1$, $\dim M_{k,n} > \dim \mathbb{R}^n$ and $\mathscr{R}_{k,n}f$ is overdetermined; the range of $\mathscr{R}_{k,n}$ was characterized as the null space of an ultra hyperbolic partial differential operator when $k = 1, n = 3$ by John [J], and by a system of such operators in general by Gelfand, Graev and Shapiro [GeGrS1]. Because of the overdeterminedness of $\mathscr{R}_{k,n}$, it is natural to ask which n-dimensional submanifolds $\mathscr{C} \subset M_{k,n}$ (called k-plane complexes) have the property that $\mathscr{R}_{k,n}f|_{\mathscr{C}}$ determines f, and to find inversion formulas giving $f(x)$ in terms of $\mathscr{R}_{k,n}f|_{\mathscr{C}}$. Even for those $\mathscr{C}$'s for which this is impossible, one can try to reconstruct as much of f as possible. The purpose of this paper is to give a partial solution to this problem in the case $k = 1$.

Already from the inversion of the full k-plane transform one knows that there is a qualitative difference in the inversion operators that depends on the parity of k. The inversion formula is (see [He])

$$f(x) = C_{k,n}(-\Delta)^{k/2}\mathscr{R}^t_{k,n}\mathscr{R}_{k,n}f(x). \tag{1.2}$$

When k is even, $(-\Delta)^{k/2}$ is a differential operator and the inversion formula is local; when k is odd, $(-\Delta)^{k/2}$ is a pseudodifferential operator and finding $f(x)$ depends on knowing $\mathscr{R}_{k,n}f(\pi)$ for all $\pi \in M_{k,n}$, and so the inversion formula is nonlocal.

The k-plane transform can also be formulated for the Grassmann bundle $M^{\mathbb{C}}_{k,n}$ of complex k-planes in $\mathbb{C}^n$, still acting on, say, smooth functions of compact support; since the underlying planes are $2k$-dimensional, one can hope for local inversion

Received November 9, 1987. Revision received May 31, 1988. First author supported in part by NSF Grant DMS-8601534. Second author supported in part by NSF Grant DMS-8601118 and an Alfred P. Sloan Research Fellowship.