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A DECOMPOSITION THEOREM FOR CERTAIN SELF-DUAL MODULES IN THE CATEGORY $\mathcal{O}$

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1. Summary. Associated to a complex, semisimple Lie algebra $\mathfrak{g}$ and a parabolic subalgebra $\mathfrak{p}_S$ is a category $\mathcal{O}_S$ containing all the generalized Verma modules induced from $\mathfrak{p}_S$ and all of their composition factors. In this paper, we classify all modules in the category $\mathcal{O}_S$ which are self-dual and have filtrations with generalized Verma modules as successive quotients. A collection of indecomposable modules with these properties is introduced first. It is then proved that any other module with these properties is isomorphic to a direct sum of modules in the collection. This generalizes a theorem proved by Enright and Shelton for certain choices of $\mathfrak{g}$ and $\mathfrak{p}_S$ with $\mathfrak{p}_S$ maximal [2]. Even in the case of the usual category $\mathcal{O}$, which corresponds to $\mathfrak{p}_S$ being a Borel subalgebra, our result is new. Results in [2] on bilinear and sesquilinear forms may also be extended to any $\mathcal{O}_S$, as discussed at the end of the paper.

2. Notation and background. Let us fix a complex, semisimple Lie algebra $\mathfrak{g}$, a Cartan subalgebra $\mathfrak{h}$, and a Borel subalgebra $\mathfrak{b}$ containing $\mathfrak{h}$. Let R be the root system and $\mathcal{W}$ the Weyl group associated to $\mathfrak{h}$. To each $\alpha \in R$ is associated a reflection s_α in $\mathcal{W}$. Let $B(R)$ be the set of simple roots of R corresponding to the choice of $\mathfrak{b}$ and let $B(\mathcal{W})$ be the corresponding set of simple reflections in $\mathcal{W}$. Let ρ be the half-sum of the positive roots. Given $w \in \mathcal{W}$ and $\mu \in \mathfrak{h}^*$, we set $w \cdot \mu = w(\mu + \rho) - \rho$. The space $\mathfrak{h}^*$ carries the usual partial order, in which the positive elements are the nonnegative, integral, linear combinations of simple roots, besides 0. Set $R_\mu = \{\alpha \in R : s_\alpha \cdot \mu \text{ and } \mu \text{ are comparable}\}$. Then R_μ is a root system, and it has a unique simple basis $B(R_\mu)$ lying in the set of positive roots of R. Let $\mathcal{W}_\mu$ be the group generated by the set $B(\mathcal{W}_\mu)$ of reflections $\{s_\alpha : \alpha \in B(R_\mu)\}$. Recall that μ is regular if $s \cdot \mu \neq \mu$ for all $s \in B(\mathcal{W})$, integral if $R_\mu = R$, and dominant if $w \cdot \mu \not> \mu$ for all $w \in \mathcal{W}$.

Associated to the choices of $\mathfrak{b}$ and $\mathfrak{h}$ is the category $\mathcal{O}$, consisting of the finitely generated $\mathfrak{g}$-modules which are $\mathfrak{h}$-semisimple and $\mathfrak{b}$-finite. Given a weight $\mu \in \mathfrak{h}^*$, the Verma module $V(\mu)$ is defined to be the $\mathfrak{g}$-module induced from the one-dimensional $\mathfrak{b}$-module on which $\mathfrak{h}$ acts via μ. It lies in $\mathcal{O}$ and has a unique simple homomorphic image $L(\mu)$. Projective covers exist in $\mathcal{O}$; we denote by $Q(\mu)$ the projective cover of $L(\mu)$. A duality functor δ is also defined on $\mathcal{O}$. In view of the central role played by δ in this paper, let us recall its definition in detail. Assume a Chevalley basis of $\mathfrak{g}$ is fixed, with each root space $\mathfrak{g}_\alpha$ spanned by an element x_α. With

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