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KHINCHIN'S INEQUALITY AND THE ZEROES OF BLOCH FUNCTIONS

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0. Introduction. H(U) will denote the space of all (complex-valued) holomorphic functions in the unit disc $\mathbf{U} \subseteq \mathbf{C}$; the Bloch space is

$$\mathscr{B}(\mathbf{U}) = \left\{ f \in H(\mathbf{U}) \colon f'(z) = O\big((1 - |z|)^{-1}\big) \right\},$$

while the "little-oh" Bloch space is

$$\mathscr{B}_o(\mathbf{U}) = \left\{ f \in H(\mathbf{U}) \colon f'(z) = o\big((1 - |z|)^{-1}\big) \right\}.$$

These are interesting objects of study at least in part because of the existence of a long list of theorems of the form "$f \in \mathscr{B}$ if and only if ..." (See [ACP], [CI]; for a new one, see [AX].) In a study of boundary behavior of functions in the unit ball of $\mathbf{C}^n$, Ahern and Rudin [AR] were led to the question, "Are the zero sets of elements of $\mathscr{B}_o$ the same as the zero sets of elements of $\mathscr{B}$?" We shall see that the answer is no:

THEOREM 2. *There exists a nonzero $f \in \mathscr{B}$ such that if $g \in \mathscr{B}_o$ and $Z(f) \subseteq Z(g)$, then $g = 0$.*

(Here $Z(f)$ is the zero set of f, "counted with multiplicity.")

Before beginning to discuss the proof of Theorem 2, let us introduce some notation. If f is a function on a probability space (X, μ), let

$$\|f\|_0 = \exp \int_X \log|f| \, d\mu \tag{1}$$

denote the geometric mean of f; note that in fact

$$\|f\|_0 = \lim_{p \to 0^+} \left(\int_X |f|^p \, d\mu \right)^{1/p}.$$

If f is a function defined in the disc and $0 \leqslant r < 1$, let $f_r(e^{i\theta}) = f(re^{i\theta})$; if g is a function defined on $\mathbf{T} = \partial \mathbf{U}$, the notation $\|g\|_p$, $\|g\|_0$ will refer to the measure $(2\pi)^{-1} \, d\theta$ on $\mathbf{T}$.

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