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A NEW TYPE OF SETS OF UNIQUENESS

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§1. Introduction. Recently many old questions in the theory of sets of uniqueness for trigonometric series have been answered using new techniques from Banach-space theory and descriptive set theory (see [9]; all definitions will be reviewed in §2). For example, Kechris and Louveau [9, chap. VIII] and then Debs and Saint Raymond [2] each gave a Borel basis for the class U_0 of sets of uniqueness in the wide sense. This fact has several important consequences ([9, chap. VIII], [2]). We shall show in §3 that the two bases are in fact the same, give a simpler proof that U_0' is indeed a basis, and unify the theory with that for U-sets and U_1-sets. This will involve some simple extensions of theorems of Banach-Dixmier and Kechris-Louveau from subspaces to convex cones in Banach spaces.

Further, less obvious, extensions of these same theorems to maps between two Banach spaces will be given in §4 to develop the theory of a new class of sets, U_2, which lies strictly between the U_1-sets and the U_0-sets. They too can be written as countable unions of special U_2-sets called U_2'-sets. The class U_2' is very natural and was briefly considered by Piateckiĭ-Shapiro [13] and, as it turns out, by the present author [11, 12] in another form. Here we establish the equivalence of the two definitions of U_2' and clarify their relations to the other types of sets of uniqueness. While this allows us to answer some previously open questions, others remain and are put into sharper relief.

The theorem of Kechris and Louveau has a further generalization to polar sets and even to conjugate convex functions. Since our main development does not require this generality, we have postponed its statement and proof until §5.

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§2. Preliminaries. In his landmark papers [13, 14], Piateckiĭ-Shapiro introduced Banach-space techniques as tools in the study of sets of uniqueness. The (complex) Banach spaces which are most important for this study are $PF(\mathbb{T})$, the space of distributions S on the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ whose Fourier coefficients $\{\hat{S}(n)\}_{n \in \mathbb{Z}}$ lie in $c_0(\mathbb{Z})$, with the $c_0(\mathbb{Z})$ norm; $A(\mathbb{T})$, the space of absolutely convergent Fourier series with the norm $\|f\|_A = \sum_{n \in \mathbb{Z}} |\hat{f}(n)|$; and $PM(\mathbb{T})$, the space of distributions with bounded Fourier coefficients under the $l^\infty(\mathbb{Z})$ norm.

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