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LIE GROUPS AND LIPSCHITZ SPACES

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1. Introduction. If $g \in G \to U(g)$ denotes a continuous representation of the connected Lie group G by bounded linear operators U acting on a Banach space $\mathscr{B}$, then an element $a \in \mathscr{B}$ satisfies a Lipschitz condition of order $\alpha \in \langle 0, 1 \rangle$ if the norm difference $\|a - U(g)a\|$ of a and its translate $U(g)a$ is of order $|g|^{\alpha}$ in a neighbourhood of the identity $e \in G$. There are several possible methods of expressing this order condition and we consider a family of Banach subspaces $\mathscr{B}_{\alpha, q}(G) \subseteq \mathscr{B}, q \in [1, \infty]$, formed by those $a \in \mathscr{B}$ for which $g \to \|a - U(g)a\|/|g|^{\alpha}$ satisfies an L_q-condition in a neighbourhood of e. These spaces are the direct analogue of the Lipschitz spaces associated with the group of translations $\mathbb{R}^n$ acting on the classical function spaces over $\mathbb{R}^n$, e.g., $L_p(\mathbb{R}^n)$, $C_0(\mathbb{R}^n)$, etc. (see, for example, [BuB], [KPS], [Ste], [Tri 1], or [Tri 2]). These original Lipschitz spaces play a variety of useful roles in the analysis of partial differential equations and it is to be expected that the $\mathscr{B}_{\alpha, q}(G)$ will be of equal utility in the analysis of the representations $(\mathscr{B}, G, U)$.

Our first aim is to prove that the main structural features of the classical Lipschitz spaces extend to the $\mathscr{B}_{\alpha, q}(G)$. For example, these spaces can be characterized as Lipschitz spaces of the associated heat semigroups, or Poisson semigroups. Alternatively, the spaces can be identified as interpolation spaces between $\mathscr{B}$ and the subspace $\mathscr{B}_1$ formed by the C_1-elements of $(\mathscr{B}, G, U)$. Our second aim is to establish that the representation of G restricted to the $\mathscr{B}_{\alpha, q}(G)$ has strong regularity properties which are not necessarily valid for the original representation, e.g., the Riesz transforms are bounded operators on the $\mathscr{B}_{\alpha, q}(G)$. These regularity properties can then be exploited to obtain new results on the structure of the original representation $(\mathscr{B}, G, U)$.

As an illustration we use the $\mathscr{B}_{\alpha, q}(G)$ to prove that the analytic elements of $(\mathscr{B}, G, U)$ coincide with the analytic elements of the Poisson semigroups. This solves a problem first raised by Goodman [Goo 1]. Goodman proved the result for unitary representations ([Goo 1], section 3) and then, with the aid of a supplementary argument of Nelson, also obtained the result for Banach space representations satisfying certain "a priori" bounds ([Goo 1], appendix). Although these latter bounds are not satisfied in general, e.g., they fail for translations on $L_1(\mathbb{R}^n)$ or $C_0(\mathbb{R}^n)$ ([Orn], [DeLM]), they are valid on the Lipschitz spaces $\mathscr{B}_{\alpha, q}(G)$, and this observation together with a simple interpolation argument solves the problem of analytic elements in general.

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