Vol. 57, No. 1 DUKE MATHEMATICAL JOURNAL © August 1988

WEIL DIVISORS AND SYMBOLIC ALGEBRAS

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Let D be a Weil divisor on a singularity $X = \operatorname{spec}(R)$, where R is a normal local domain. We are interested in determining which X satisfy the following condition:

(∗) For every Weil divisor D on X, $\oplus_{n \geqslant 0} \mathcal{O}_X(nD)$ is finitely generated as an $\mathcal{O}_X$ algebra.

More generally, one can ask:

(∗∗) For a given Weil divisor D on X, is $\oplus_{n \geqslant 0} \mathcal{O}_X(nD)$ finitely generated as an $\mathcal{O}_X$ algebra?

In other words, (∗) says that the symbolic algebra $\oplus I^{(n)}$ is finitely generated for every height-one unmixed ideal $I \subset R$. (∗∗) asks if $\oplus I^{(n)}$ is finitely generated for a given height-one unmixed ideal $I \subset R$.

In dimension 2, (∗∗) is true for a divisor D if and only if some multiple nD is Cartier. This is not true in higher dimension.

Besides their intrinsic interest, the problems (∗) and (∗∗) are of importance because they are fundamental in understanding higher-dimensional algebraic geometry. For instance, Kawamata [9] shows that a class of singularities satisfy (∗) to prove the existence of minimal models of semistable degenerations of surfaces. Mori [18] solves (∗∗) in some special cases to prove the minimal model conjecture in dimension 3.

Lipman [12] showed that (∗) characterizes rational surface singularities $X = \operatorname{spec}(R)$ when R is a normal 2-dimensional Henselian local domain. Since there exist examples of nonrational singularities which are factorial in higher dimension, this property cannot characterize rational singularities there. However, we have the following necessary condition for X to satisfy (∗). Let R be a normal analytic local domain with a resolution of singularities $f\colon Y \to X = \operatorname{specan}(R)$. We show that if X satisfies (∗), then $R^1 f_* \mathcal{O}_Y = 0$. If R is algebraizable, this implies that R is S_3. This is known when R is factorial [6].

A recent result of Kawamata [9] shows that in dimension 3, a large class of rational singularities, including all Gorenstein rational singularities, do satisfy (∗). It is then natural to ask whether every rational singularity has this property. We give an example to show that the answer is no. The example is a rational log-canonical singularity, which is the simplest type of singularity for which one

Received April 23, 1987.