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ON ISOSPECTRAL PERIODIC POTENTIALS ON A DISCRETE LATTICE I

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§1. Introduction. In the "nearly free electron approximation" for the description of the behavior of the conduction electrons in a crystal one assumes that the strong interaction of these electrons with each other and with the positive ions have the net effect of a very weak D-periodic potential Q, where D denotes a certain lattice in $\mathbf{R}^3$ (cf. [AM]). The eigenstates Ψ of a conduction electron can then be described by the solutions of $(-\Delta + Q)\Psi = \lambda\Psi$ with $\Psi(x + \ell) = e^{i\alpha\cdot\ell}\Psi(x)$ for all vectors $\ell = (\ell_1, \ell_2, \ell_3)$ in the lattice D. The vector $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is called the crystal momentum. It is a fact that many properties of a crystal depend only on the family of spectra (Bloch spectrum) of $-\Delta + Q$, parametrized by the values of α, which enters the boundary condition. Observe that the periodic spectrum corresponds to the value $\alpha = 0$. We are interested in the following problem: To what extent is Q determined by one or by the whole family of spectra described above? A number of partial answers to this problem are known.

For $d = 1$ the situation is well understood [McKT]. The Bloch spectrum is determined by the periodic spectrum. The set of potentials with the same Bloch spectrum is generically an infinite-dimensional torus (see [McKT]). There is a number of results concerning the Schrödinger equation on compact Riemannian manifolds without boundary. For certain manifolds the spectrum of $-\Delta + Q$ does determine Q uniquely up to symmetries. For others there are counterexamples. Closely related are results concerning the question to what extent the metric of a given manifold is determined by the spectrum of the Laplacian (cf. e.g., [B], [Br], [D], [Go1], [Go2], [DG], [GW1], [GW2], [GW3], [G], [GK1], [GK2], [I], [S], [V]).

In this paper we look at a discretized and generalized version of the Schrödinger equation. To be more specific, let us introduce the following notation: Let D denote a lattice in $\mathbf{Z}^d$ whose fundamental domain is given by $\Gamma = \{x = (x_1, \ldots, x_d) \in \mathbf{Z}^d\colon 1 \leqslant x_i \leqslant p_i\}$, where $p_i \geqslant 2$ are integers. Then we look at the following eigenvalue problem:

$$(1) \qquad (L + Q)u = \lambda u$$

$$(2) \qquad u(x + p_j e_j) = e^{i\alpha_j} u(x) \; (x \text{ in } \mathbf{Z}^d, 1 \leqslant j \leqslant d)$$

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