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CONJUGACY CLASSES OF n-TUPLES IN LIE ALGEBRAS AND ALGEBRAIC GROUPS

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Let G be a reductive algebraic group over an algebraically closed field F of characteristic zero and let $L(G)$ be the Lie algebra of G. Then G acts on G by inner automorphisms and it acts on $L(G)$ by the adjoint representation. By taking the diagonal actions, we get actions of G on the spaces of n-tuples G^n and $L(G)^n$. In this paper we will prove a number of geometric properties of the orbits of G on these spaces of n-tuples.

For $n = 1$ the situation has been studied in great detail (see [12, 30]). For example, if $x \in G$, then the orbit $G \cdot x$ is closed (resp. unstable) if and only if x is semisimple (resp. unipotent). Let x have Jordan decomposition $x = su$, with s semisimple and u unipotent. Then the stabilizer G_x is the intersection of G_s and G_u and $G \cdot s$ is the unique closed orbit in the closure of $G \cdot x$. Let V be the affine variety corresponding to the algebra $F[G]^G$ of regular class functions on G and let π: $G \to V$ be the morphism of affine varieties corresponding to the inclusion homomorphism $F[G]^G \to F[G]$. Then each fibre $\pi^{-1}(v)$, $v \in V$, has codimension equal to the rank of G.

All of these results, plus a number of others, can be generalized to the action of G on n-tuples. For now, we will restrict our discussion to the action of G on G^n. Let $\mathbf{x} = (x_1, \ldots, x_n) \in G^n$ and let $A(\mathbf{x})$ be the algebraic subgroup of G generated by $\{x_1, \ldots, x_n\}$. We say that $\mathbf{x}$ is a *semisimple n-tuple* (resp. *unipotent n-tuple*) if $A(\mathbf{x})$ is a linearly reductive (resp. unipotent) algebraic group. We show that the orbit $G \cdot \mathbf{x}$ is closed (resp. unstable) if and only if $\mathbf{x}$ is a semisimple (resp. unipotent) n-tuple. Let L be a Levi subroup of $A(\mathbf{x})$. Then each x_i can be written uniquely in the form $x_i = y_i z_i$, with $y_i \in L$ and $z_i \in R_u(A(\mathbf{x}))$. Let $\mathbf{y} = (y_1, \ldots, y_n)$ and let $\mathbf{z} = (z_1, \ldots, z_n)$. The decomposition $\mathbf{x} = \mathbf{y} \cdot \mathbf{z}$ is called a Levi decomposition of $\mathbf{x}$. (Such a decomposition of n-tuples is not unique.) We show that the stabilizer $G_{\mathbf{x}}$ is the intersection of $G_{\mathbf{y}}$ and $G_{\mathbf{z}}$ and that $G \cdot \mathbf{y}$ is the unique closed orbit in the closure of $G \cdot \mathbf{x}$. Let G^n/G be the affine variety corresponding to the algebra $F[G^n]^G$ of invariants and let π: $G^n \to G^n/G$ be the morphism corresponding to the inclusion homomorphism $F[G^n]^G \to F[G^n]$. Let $\mathbf{x} \in G^n$ be a semisimple n-tuple and let A be a maximal torus of the stabilizer $G_{\mathbf{x}}$. Then the dimension of the fibre $\pi^{-1}(\pi(\mathbf{x}))$ depends only on the G-conjugacy class of A and we give a precise formula for this dimension. (For $n = 1$, A is a maximal torus of G.) We also characterize the stable points of G^n and the smooth points of the quotient variety G^n/G.

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