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A SHARP INEQUALITY FOR THE SQUARE FUNCTION

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1. Introduction. In his Colloquium Lectures on the uncertainty principle [2], C. Fefferman has made the following statement: "This leads to deep questions of Fourier analysis, such as those treated by R. Fefferman." The *this* in question is the study of the square function, and in particular of the following problem: Let $\tilde{M}$ be a homogeneous, positive operator. When is it true that

$$\int |f^*|^2 V\,dx \leqslant C(\tilde{M}) \int S^2(f) \tilde{M} V\,dx \tag{1}$$

for all nonnegative V in $L^1_{\text{loc}}(\mathbf{R}^d)$ and all $f \in \mathscr{C}_0^\infty(\mathbf{R}^d)$? In this paper we characterize, to essentially best possible, those $\tilde{M}$ for which (1) holds.

Let us define our terms. For $Q \subset \mathbf{R}^d$ a dyadic cube, we let $\ell(Q)$ denote its sidelength and $|Q|$ its Lebesgue measure. (Henceforth Q will always denote a cube, and all cubes are assumed to be dyadic.) For $f \in L^1_{\text{loc}}(\mathbf{R}^d)$ define

$$f_Q = \frac{1}{|Q|} \int_Q f,$$

and for k an integer set

$$f_k = \sum_{\ell(Q) = 2^{-k}} f_Q \chi_Q.$$

For $\ell(Q) = 2^{-k}$ define

$$a_Q(f) = (f_{k+1} - f_k)\chi_Q.$$

Clearly,

$$\begin{aligned} f &= \lim_{k \to \infty} f_k \\ &= \sum_Q a_Q(f), \end{aligned}$$

a.e. and in L^2, for any "reasonable" f (e.g., $f \in \mathscr{C}_0^\infty(\mathbf{R}^d)$). We define the dyadic

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