Vol. 55, No. 2 DUKE MATHEMATICAL JOURNAL © June 1987

ON SCHOTTKY AND KOEBE-LIKE UNIFORMIZATIONS

DENNIS A. HEJHAL

This note is an elaboration and extension of [11, section 2]. For an overview of the contents, consult theorems 1, 2, 4, 6.

I. Circle domains. It is well known that any smoothly bounded plane domain of finite connectivity is *conformally equivalent* to a domain bounded by circles, and that any such (conformal) mapping is unique up to an auxiliary linear fractional transformation.

This result is due to P. Koebe [18–21]. Cf. [10, pp. 201–217] for a self-contained treatment based on the time-honored continuity method.

II. A generalization of the Koebe mapping. Given any positive integer $p \geqq 3$. Let $\mathscr{R}_0$ denote the Riemann surface of $u = z^{1/2}$ regarded as a ramified covering of $\hat{\mathbf{C}}$. There are two sheets with branch points located at $z = 0, \infty$. For the sake of definiteness: we take the branch cut to be $\{z \in \hat{\mathbf{C}} : \arg(z) = \pi\}$. Cf. figure 1. Let $\mathscr{R} = \mathscr{R}_0 - \{\text{the branch points}\}$.

Let Φ be the holomorphic function on $\mathscr{R}$ defined by writing

$$\Phi(z) = \Phi[(z, u)] = \mathrm{u}.$$

Loosely speaking: Φ is the branch of $\sqrt{z}$ which takes the value $+1$ at the point $z = 1$ in the upper sheet of $\mathscr{R}$. A moment's thought shows that Φ is a 1-1 conformal mapping of $\mathscr{R}$ onto $\hat{\mathbf{C}} - \{0, \infty\}$.

Let $\mathscr{B}$ be the family of domains (of connectivity p) depicted by figure 2. The paths $C_0, C_1, \ldots, C_{p-1}$ are circles $|z - a_j| = r_j$ subject to the condition $a_0 = a_1 = 0$, $r_0 < a_2 < 1$, $r_1 = 1$.

Let $\mathscr{M}$ be the family of circle domains on $\mathscr{R}$ depicted by figure 3. The paths $\Gamma_0, \Gamma_1, \ldots, \Gamma_{p-1}$ are circles $|z - b_j| = R_j$ where we assume that:

$b_0 = b_1 = 0$, $R_0 < b_2 < 1$, $R_1 = 1$

Γ_0 and Γ_1 go around twice

$\Gamma_2, \ldots, \Gamma_{p-1}$ go around once

Γ_2 is located in the first sheet (i.e., Φ has positive real part).

Received January 31, 1986. Revision received October 6, 1986. Supported in part by NSF Grant MCS 83-03535. This paper was written while the author was a visiting member of the Institute for Advanced Study.