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ON LIMIT MULTIPLICITIES OF REPRESENTATIONS WITH COHOMOLOGY IN THE CUSPIDAL SPECTRUM

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Introduction. Let G be a connected noncompact semi-simple Lie group with maximal compact subgroup K and denote by V an irreducible finite dimensional representation of G. If $\Gamma \subset G$ is an arithmetic subgroup it is well known that the abstract group cohomology $H^{\cdot}(\Gamma, V)$ of Γ with coefficients V coincides with the $(\mathfrak{g}, K)$-cohomology $H^{\cdot}(\mathfrak{g}, K, C^{\infty}(\Gamma \setminus G) \otimes V)$ where $\mathfrak{g}$ is the Lie algebra of G and $C^{\infty}(\Gamma \setminus G)$ the space of C^{∞}-functions on $\Gamma \setminus G$. Let $L^2_{\mathrm{cusp}}(\Gamma \setminus G)$ be the space of cusp forms in the space $L^2(\Gamma \setminus G)$ of square integrable functions on $\Gamma \setminus G$. We denote the natural image of $H^{\cdot}(\mathfrak{g}, K, L^2_{\mathrm{cusp}}(\Gamma \setminus G) \otimes V)$ in $H^{\cdot}(\Gamma, V)$ by $H^{\cdot}_{\mathrm{cusp}}(\Gamma, V)$.

In this paper we assume throughout that $\Gamma \setminus G$ is not compact and consider the behavior of $H^{\cdot}_{\mathrm{cusp}}(\Gamma, V)$ when Γ becomes smaller and smaller.

To make our statements precise we use the notion of a tower of arithmetic subgroups which is a collection $\{\Gamma_i\}_{i \in \mathbb{N}}$ of arithmetic subgroups of G with $\Gamma_{i+1} \subset \Gamma_i$ for all i, $\bigcap_{i=0}^{\infty} \Gamma_i = \{1\}$ such that all Γ_i's are normal in $\Gamma = \Gamma_0$.

The following results have been announced in [R-Sp].

THEOREM. *If $\{\Gamma_i\}_{i \in \mathbb{N}}$ is a tower of arithmetic subgroups, then*

$$\lim_{i \to \infty} [\Gamma : \Gamma_i]^{-1}\big(\dim H^{\cdot}(\Gamma_i, V) - \dim H^{\cdot}_{\mathrm{cusp}}(\Gamma_i, V)\big) = 0.$$

This is proved in §2 where we treat the contributions of the Borel–Serre boundary and in §3 where we handle residual contributions.

We write as usual $\chi(\Gamma_i, V)$ resp. $\chi_{\mathrm{cusp}}(\Gamma_i, V)$ for the Euler–Poincaré characteristic computed from $H^{\cdot}(\Gamma_i, V)$ resp. $H^{\cdot}_{\mathrm{cusp}}(\Gamma_i, V)$ and if $V = \mathbb{C}$, we omit the letter V in the notion of Euler–Poincaré characteristics.

We get as an easy application:

COROLLARY. *If $\{\Gamma_i\}_{i \in \mathbb{N}}$ is a tower of arithmetic subgroups, then*

$$\lim_{i \to \infty} [\Gamma : \Gamma_i]^{-1} \chi_{\mathrm{cusp}}(\Gamma_i, V) = \chi(\Gamma) \dim V$$

where in case that Γ has torsion elements $\chi(\Gamma)$ is the virtual Euler Poincaré characteristic, see [*S*].

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