Vol. 55, No. 1 DUKE MATHEMATICAL JOURNAL © March 1987

# A RELATION BETWEEN DEDEKIND SUMS AND KLOOSTERMAN SUMS

ILAN VARDI

**1. Introduction and statement of results.** The Dedekind sum is defined for relatively prime integers $c$, $d$ by:

$$s(d,c) = \sum_{j=1}^{c} ((j/c)) \cdot ((jd/c)),$$

where $((x)) = \langle x \rangle - \frac{1}{2}$ for $x$ a real number, and $\langle x \rangle \in [0,1)$ is the fractional part of $x$.

The classical Kloosterman sum is:

$$S(m,n,c) = \sum_{\substack{0<d<c \\ (d,c)=1 \\ ad\equiv 1 (\mathrm{mod}\ c)}} e\left(\frac{am+dn}{c}\right),$$

where $e(x) = e^{2\pi i x}$.

We prove a simple relation between these sums:

THEOREM 1.1. *Let $m \in N$, then*:

$$\sum_{\substack{0<d<c \\ (d,c)=1}} e(12ms(d,c)) = S(m,m,c).$$

*Proof.* It is known [Rademacher] that $12s(d,c) - \dfrac{a+d}{c}$ is always an integer whenever $ad \equiv 1 (\mathrm{mod}\ c)$. So we have:

$$\sum_{\substack{0<d<c \\ (d,c)=1}} e(12ms(d,c)) = \sum_{\substack{0<d<c \\ (d,c)=1 \\ ad\equiv 1 (\mathrm{mod}\ c)}} e\left(\frac{am+dm}{c}\right) = S(m,m,c). \qquad \blacksquare$$

As an application of this, we have

THEOREM 1.2.

$$\sum_{0<c<x} \sum_{\substack{0<d<c \\ (d,c)=1}} e(12ms(d,c)) \ll x^{\frac{3}{2}+\varepsilon}, \quad \forall \varepsilon > 0. \tag{1.1}$$

Received May 27, 1986. Revision received September 26, 1986.