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CONCERNING A CONJECTURE OF COLLIOT-THÉLÈNE AND SANSUC

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1. Introduction. When X is a rational surface over a field k that is not algebraically closed, the group $A_0(X)$ of zero cycles of degree zero modulo rational equivalence can be quite difficult to calculate. Much insight was gained when Bloch [1] and Colliot-Thélène and Sansuc [3] defined a homomorphism

$$\Phi\colon A_0(X) \to H^1(k, S)$$

where S is the torus $\operatorname{Hom}_{\mathbf{Z}}(\operatorname{Pic}(\overline{X}), \overline{k}^*)$, X is smooth and projective over k, and k is perfect. When k is a local or global field, it was shown in [1] that the image of Φ is finite, and in [2] that Φ is an injection, so $A_0(X)$ itself is finite.

When k is a number field, we have the commutative diagram

$$\begin{array}{ccccccc} 0 \to & Ш(A_0(X)) & \to & A_0(X) & \to & \coprod_v A_0(X_{k_v}) \\ & & & \downarrow \Phi & & \downarrow \Phi_v \\ 0 \to & Ш^1(k, S) & \to & H^1(k, S) & \to & \coprod_v H^1(k_v, S) \end{array}$$

where v ranges over all places of k. There is an induced injection

$$\phi\colon Ш(A_0(X)) \to Ш^1(k, S).$$

In [3] it was conjectured that this is in fact an isomorphism. It is this conjecture that we wish to examine.

The conjecture is clearly true whenever $Ш^1(k, S) = 0$. From [10], we know that $Ш^1(k, S)$ is dual over $\mathbf{Q}/\mathbf{Z}$ to $Ш^2(k, \operatorname{Pic}(\overline{X}))$, which is defined to be the kernel of

$$H^2(k, \operatorname{Pic}(\overline{X})) \to \coprod_v H^2(k_v, \operatorname{Pic}(\overline{X}))$$

where v ranges over the places of k. If $G = \operatorname{Gal}(\overline{k}/k)$, then $Ш^2(k, \operatorname{Pic}(\overline{X}))$ injects into $Ш^2_\omega(G, \operatorname{Pic}(\overline{X}))$, which is defined to be the kernel of

$$H^2(G, \operatorname{Pic}(\overline{X})) \to \coprod_{g \in G} H^2(\langle g \rangle, \operatorname{Pic}(\overline{X})).$$

Consequently, we see that the conjecture follows easily whenever $Ш^2_\omega(G, \operatorname{Pic}(\overline{X})) = 0$.

One class of rational surfaces whose geometry is particularly well understood is the class of nonsingular Del Pezzo surfaces. It is well known that if X is a Del Pezzo surface of degree greater than 4 over a perfect field k, then $\operatorname{Pic}(\overline{X})$ is stably

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