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CONGRUENCES FOR PERIODS OF MODULAR FORMS

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To Yuri Ivanovich Manin on the occasion of his fiftieth birthday

§1. Introduction. Starting with Yu. I. Manin's paper [17], much attention has been devoted to p-adic congruences for the periods of modular forms of varying weights k. Our first purpose is to give an exposition of a part of this work. In addition, we give a congruence of the Ramanujan type for the odd periods of a cusp form for $SL_2\mathbf{Z}$, in the six cases when the space of such cusp forms is 1-dimensional (Theorem 2). Finally, in §4 we discuss how p-adic congruences can be used as evidence of the existence of canonical square roots of the central critical Hecke L-series values $L(\psi^{k-1}, k/2)$, presumably connected with the arithmetic of the corresponding elliptic curve.

§2. Periods of cusp forms for $SL_2(\mathbf{Z})$. Let $\Phi(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$ be an element of the space S_k of cusp forms of weight k for the full modular group $SL_2(\mathbf{Z})$. The moments or *periods* of Φ are defined as follows:

$$r_m(\Phi) = \int_0^{i\infty} \Phi(z) z^m \, dz, \qquad m \geqslant 0. \tag{1}$$

Set $w = k - 2$. The periods of Φ for $m \leqslant w$ are closely related to the values at integer points in the critical strip of the Hecke L-series corresponding to Φ, which is defined in a right half-plane by $L_\Phi(s) = \sum a_n n^{-s}$. This will be described more precisely later. These L-series values are called "special values" or "critical values" of the L-function.

Eichler–Shimura's rationality result. If the coefficients a_n are real, then the periods $r_m(\Phi)$ with m odd are real, and the even periods are purely imaginary. One then separately considers the $(w/2)$-tuple $\mathbf{r}^-(\Phi)(r_1(\Phi), \ldots, r_{w-1}(\Phi)) \in \mathbf{R}^{w/2}$ of odd periods and the $(w/2+1)$-tuple $\mathbf{r}^+(\Phi) \in \mathbf{R}^{w/2+1}$ of even periods. Eichler and Shimura found a set of linear equations with rational integer coefficients which are satisfied by these period vectors for any $(\frac{1}{i}r_0(\Phi), \ldots, \frac{1}{i}r_w(\Phi))\Phi \in S_k$. Moreover, if $V^- \subset \mathbf{R}^{w/2}$ and $V^+ \subset \mathbf{R}^{w/2+1}$ denote the subspaces consisting of all vectors which satisfy these linear relations, then Eichler–Shimura proved that the map $S_k \to V^-$ given by $\Phi \mapsto \mathbf{r}^-(\Phi)$ is an isomorphism, and the map $S_k \to V^+$ given by $\Phi \mapsto \mathbf{r}^+(\Phi)$ is an isomorphism onto a subspace of codimension 1 in V^+. For a complete exposition, see [13].

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