Vol. 54, No. 1 DUKE MATHEMATICAL JOURNAL © 1987

CYCLES, CURVES AND VECTOR BUNDLES ON AN ALGEBRAIC SURFACE

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To Yu. I. Manin on his 50th birthday

Introduction. Let X be a smooth algebraic surface. ξ an effective 0-cycle on X and $\mathscr{D}$ a divisor class. If the ideal sheaf J_ξ satisfies $h^1(J_\xi(\mathscr{D})) \neq 0$, then Serre duality and the Ext construction give a torsion-free sheaf $E(\xi, \mathscr{D})$ on X. Geometrical properties of ξ and $\mathscr{D}$ can be described in terms of those of $E(\xi, \mathscr{D})$ and vice versa. For example, if $\deg \xi = 1$ (or 2) and $h^1(\mathscr{O}_X(\mathscr{D})) = 0$, then $h^1(J_x(\mathscr{D})) > 0$ (or $h^1(J_{x,y}(\mathscr{D})) > 0$) means that x is a fixed point of the complete linear system $|\mathscr{D}|$ (respectively that the points x and y are not distinguished by the curves of $|\mathscr{D}|$), and so on. Using simple properties of $E(x, \mathscr{D})$, $E(x + y, \mathscr{D})$, one can get, for example, all the results on complete linear systems on a $K3$ surface, and can prove following I. Reider [8] that $|2K_x|$ has no fixed points on an algebraic surface of general type X with $K_X^2 \geqslant 5$, and so on. Conversely, by the description of fixed points of the complete linear system $|3K_X|$ on the Severi-Dolgachev surface $X = F_{2,3}$, S. Donaldson determined the structure of the moduli space of stable rank 2 vector bundles on $F_{2,3}$ with $c_1 = 0$, $c_2 = 1$.

In our earlier paper [10] we used this method to describe the variety of special cycles on a polarized $K3$ surface. In the present paper we apply this construction for the description of

(i) the structure of the moduli space of simple vector bundles on a $K3$ surface (§4, Ch. II);

(ii) the specific character of Brill-Noether theory for smooth curves on a $K3$ surface (R. Lazarsfeld's construction) (§5, Ch. II);

(iii) the special case of the infinitesimal Mumford-Harris conjecture (§6, Ch. II).

In Chapter I we collect together the simplest properties of the construction $(\xi, \mathscr{D}) \mapsto E(\xi, \mathscr{D})$ on a regular algebraic surface X.

We use the standard notation of [2] and [3].

Chapter I. The construction

§1. Cycles and divisor classes. On a smooth algebraic surface X which is regular ($h^1(\mathscr{O}_X) = 0$), a 0-dimensional subscheme $\xi \subset X$ of length d is called a *cycle* of degree d on X. Each cycle is defined by the ideal sheaf $J_\xi \subset \mathscr{O}_X$ or by the

Received July 15, 1986.