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# COHOMOLOGY OF COMPACTIFICATIONS OF ALGEBRAIC GROUPS

C. DE CONCINI AND C. PROCESI

**Introduction.** Let $G$ be a semisimple complex algebraic group of adjoint type. In this paper we compute the rational cohomology ring of a class of smooth compactifications $\bar{G}$ of $G$ (which we call regular); these compactifications are equivariant with respect to the left and right action of $G$.

We fix a maximal torus $T$ of $G$ and denote by $W$ its Weyl group.

For each $G \times G$ equivariant compactification $\bar{G}$ of $G$ the closure $\bar{T}$ of $T$ in $\bar{G}$ is a torus embedding with an induced action of $W$.

In [6] we have studied the basic compactification $\bar{G}^0$ which corresponds, in the case of the projective linear group, to the complete projectivities of classical enumerative geometry [16].

In [7] (Theorem 5.3) we have shown that the torus embedding $\bar{T}^0$, lying into $\bar{G}^0$, is the one associated to the decomposition into Weyl chambers of $\hom(X(T), R)$ ($X(T)$ denoting the character group of $T$) according to the combinatorial theory of torus embeddings [10].

In [7] (Theorem 5.2) we have also proved that, by associating to $\bar{G}$ the torus embedding $\bar{T}$, one establishes a bijection between torus embeddings, lying over $\bar{T}^0$ and with $W$ action, and $G \times G$ equivariant embeddings of $G$ lying over $\bar{G}^0$.

In both the previously quoted papers we treat in fact the more general case of symmetric varieties.

Let now $K$ be a maximal compact subgroup of $G$ such that $T_K = T \cap K$ is a maximal compact torus; in this paper we show that the classical Cartan decomposition $G = KTK$ is the clue to the cohomology theory of the varieties $\bar{G}$.

In fact, given $\bar{G}$ and $\bar{T}$ as before, we have a Cartan decomposition

$$\bar{G} = K\bar{T}K.$$

More geometrically since $K \times K$ acts on $\bar{G}$ and $\bar{T}$ is $T_K \times T_K$ stable we have a map

$$\pi\colon (K \times K) \times_{T_K \times T_K} \bar{T} \to \bar{G}.$$

The manifold $(K \times K) \times_{T_K \times T_K} \bar{T}$ is in fact an algebraic variety which can be considered as a "relative torus embedding" over $K/T_K \times K/T_K$ (the product of the flag variety with itself).

The Weyl group $W$ acts naturally on $(K \times K) \times_{T_K \times T_K} \bar{T}$ and $\pi$ is constant on $W$ orbits.

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