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DEGENERATIONS OF QUINTIC THREEFOLDS AND THEIR LINES

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Introduction. Let V be a generic quintic threefold in $\mathbf{P}^4$. Then it is known that V contains exactly 2875 lines [6].

Given a generic pencil $\{V_t\}_{t \in \mathbf{P}^1}$ of quintic threefolds, with V_0 a transversally intersecting union of two components, and smooth for t in a punctured disk Δ^*, the associated family $\mathscr{L}_t$ of lines on V_t abstractly forms a 2875 sheeted covering of Δ^*; the closure will give a (possibly branched) covering of Δ. In this paper, the geometry of the limiting lines $\mathscr{L}_0$ on V_0 will be described.

I'd like to take the opportunity here to thank Herb Clemens for helpful conversations.

1. First order deformations of rational curves. We start with some terminology and notation.

Definition. Let $\{V_t\}_{t \in \Delta}$ be a family of algebraic varieties, and $W \subset V_0$ a subvariety. We say that W *deforms with* $\{V_t\}$ if there is a proper, flat family $\{W_t\}_{t \in \Delta'}$, $0 \in \Delta' \subset \Delta$, with $W_0 = W$. Note that with this definition, it is not allowed that W fits into a family parameterized by a branched cover of Δ; in other words, monodromy may prevent W from deforming with $\{V_t\}$.

In the situation of the introduction, the equation of V_t may be written as

$$GH + tF = 0 \tag{1.1}$$

where G, H are equations for the two components of V_0, and F is an equation for a general element of the pencil. Let g, h be the degrees of G, H respectively, so that $g + h = 5$. We will frequently abuse notation by using the same letter to denote an equation or its zero locus.

In terms of this pencil, the problem under consideration may be related to the following question: Given a line $L \subset V_0$, does L deform with $\{V_t\}$?

We consider the possible manners in which L lies in V_0.

(a) $L \subset G \cap H$. Since $G \cap H$ is a generic $K3$ surface, this possibility does not occur.

(b) $L \subset G$, $L \not\subset H$

(b1) if $\{g, h\} = \{2, 3\}$, then L lies on a quadric (3 dimensions of such L for a fixed quadric) or a cubic (2 dimensions).

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