Vol. 50, No. 3 DUKE MATHEMATICAL JOURNAL © September 1983

HILBERT TRANSFORMS FOR CONVEX CURVES

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§0. Introduction. Let $\Gamma : \mathsf{R} \to \mathsf{R}^n$ be a curve in R^n, $n \geqslant 2$. To Γ we associate the Hilbert transform operators $\overline{H}$ and H given by the principal value integrals

$$\overline{H}f(x) = \int_{-1}^{1} f(x - \Gamma(t))\,\frac{dt}{t} \qquad (x \in \mathsf{R}^n) \tag{1}$$

$$Hf(x) = \int_{-\infty}^{\infty} f(x - \Gamma(t))\,\frac{dt}{t} \qquad (x \in \mathsf{R}^n). \tag{2}$$

(Here and throughout, f will denote an arbitrary function in an appropriate class; say, $f \in C_c^\infty(\mathsf{R}^n)$.) A great deal of attention has been given to the problem of determining for which curves Γ and for what values of p we have the L^p bounds

$$\|\overline{H}f\|_p \leqslant A_p \|f\|_p \tag{3}$$

$$\|Hf\|_p \leqslant A_p \|f\|_p \tag{4}$$

for constants A_p depending only on p and Γ. (Part I of [SW2] contains a useful survey of these and other problems, and an extensive list of references. More recent results can be found in [Ne], [NSW], and [We].)

It is known that if Γ has a sufficient amount of curvature, then (3) holds for $1 < p < \infty$. (For example, if the derivatives of $\Gamma(t)$ elevated at $t = 0$ span R^n, then (3) holds for $1 < p < \infty$. See [NRW] and [SW2].)

On the other hand, it was shown in [NW] that in the case of plane curves (i.e., $n = 2$), (3) may fail even when $p = 2$ and Γ is of the form $\Gamma(t) = (t, \gamma(t))$, with $\gamma(t)$ a smooth, odd curve which for $t > 0$ is positive, increasing, and convex. (However, it was demonstrated there that if also $\gamma''(t)$ is increasing for $t > 0$, then $\|Hf\|_p \leqslant A_p \|f\|_p$ for $\frac{5}{3} < p < \frac{5}{2}$.)

The main object of this paper is to give a simple necessary and sufficient condition on curves of the above type so that $\|Hf\|_2 \leqslant A\|f\|_2$ and $\|\overline{H}f\|_2 \leqslant A\|f\|_2$. The condition is expressed in terms of an auxilliary function

$$h(t) = t\gamma'(t) - \gamma(t). \tag{5}$$

[1] Received September 25, 1982. Supported in part by an NSF grant at the University of Wisconsin.