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THE UNIFORM CENTRAL LIMIT THEOREM FOR THETA SUMS

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0. Introduction. Since the sequence of Rademacher functions, $r_n(x) = \operatorname{sgn}\sin(2^n\pi x)$, are independent as random variables on $[0,1]$, their normalized sums are asymptotically normally distributed according to the central limit theorem. Thus if $R_N(x) = \sum_{n=1}^{N} r_n(x)/\sqrt{N}$ and $E_N(\lambda) = \{x \text{ real: } \lambda \leqslant R_N(x)\}$ for λ real, then $|[0,1] \cap E_N(\lambda)| \to \mathscr{L}(\lambda)$ as $N \to \infty$, where $|\cdot|$ denotes Lebesgue measure and $\mathscr{L}(\lambda) = (1/\sqrt{2\pi})\int_\lambda^\infty \exp(-x^2/2)\,dx$, the normal distribution. Further, it is easy to show, again using independence, that a uniform central limit theorem holds in that for any $a < b$, $|[a,b] \cap E_N(\lambda)| \to (b-a)\mathscr{L}(\lambda)$.

Recently we considered the sequence $\exp(i\pi n^2 x)$, which are not independent, and the sums $S_N(x) = \sum'^{N}_{n=0} \exp(i\pi n^2 x)$ for $N \geqslant 1$ real, where the prime indicates that the terms corresponding to $n = 0$ and $n = N$, if N is an integer, carry a factor of $1/2$. We let $E_N(\lambda) = \{x \geqslant 0 : \lambda \leqslant |S_N(x)|/\sqrt{N}\}$ for $\lambda \geqslant 0$ and showed that as N tends to ∞, $|[0,1] \cap E_N(\lambda)|$ converges to a limit, $\Phi(\lambda)$, at points of continuity of the limit [5]. The limit is not normal, and the methods used to show convergence were not probabilistic but were related to the circle method of number theory. Now we are able to show a uniform central limit theorem for S_N, namely that if $0 \leqslant a \leqslant b$ then $|[a,b] \cap E_N(\lambda)| \to (b-a)\Phi(\lambda)$. The proof of this is different from the proof in [5] and requires an argument involving an estimate of Kloosterman sums. Thus the extension of the central limit theorem from $[0,1]$ to $[a,b]$ is much more difficult for S_N than for R_N.

The methods use an asymptotic transformation formula which describes S_N about certain rational numbers in terms of one of three Fourier series. The measure of $[a,b] \cap E_N(\lambda)$ is transformed to a sum across rationals of measures of subsets. The condition $a \leqslant x \leqslant b$ translates into a characteristic function of an interval modulo 1. These characteristic functions we approximate by trigonometric series which reduces the calculation to estimating Kloosterman sums. Then we pass to the limit, where the argument is similar to [4] and [5]. We use these methods also to show a parallel theorem for the theta series $\Theta(\sigma + it) = \sum_{n=-\infty}^{\infty} \exp(-\pi n^2(\sigma + it))$.

As an application we have calculated the α-th moments of $|S_N|$ for $0 < \alpha < 4$ asymptotically as $N \to \infty$ on arbitrary subintervals of $[0,\infty)$. Naturally these are also equal to the length of the subinterval times the moment over $[0,1]$. By other means we have calculated the higher moments and have found that only for

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