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FAMILIES OF ZARISKI SURFACES

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Introduction. This note is a first step in the analysis of the moduli of uni-rational surfaces in characteristic $p > 0$. We examine the behavior of the so-called generalized Zariski surfaces under smooth specialization, and show that this class of surfaces is closed under this operation. We also describe the structure of the totality of all generalized Zariski surfaces in a fixed projective space, and show that this set is a countable union of closed subsets of the Hilbert scheme of smooth surfaces in the projective space. The analogous results also hold for the class of Zariski surfaces. Finally, we give an example of a deformation of a Zariski surface to a smooth surface that is not uni-rational, or even uni-ruled.

We will use the following notations and conventions:

If $f: X \to M$ is a morphism of schemes, and m is a point of M, we let X_m denote the scheme-theoretic fiber $f^{-1}(m)$. We let $|X_m|$ denote the reduced scheme associated to X_m. If $g: Y \to M$ is another morphism of schemes, and $F: X \to Y$ is an M-morphism, we let F_m denote the morphism

$$F_m : |X_m| \to |Y_m|.$$

If G is a sheaf on X, we let G_m denote the induced sheaf on X_m. If $h: C \to M$ is on M-scheme, we let X_C denote the fiber product $X \times_M C$.

Let Q be a polynomial with rational coefficients, $f: X \to M$ a flat and projective morphism. We let $\mathrm{Hilb}^Q_{X/M}$ denote the part of the relative Hilbert scheme, $\mathrm{Hilb}_{X/M}$, of X over M that corresponds to the polynomial Q. We let $H^Q_{X/M}$ denote the universal family of subschemes of X parametrized by $\mathrm{Hilb}^Q_{X/M}$. We will often omit the subscript X/M if there is no cause for confusion.

We fix at the outset an algebraically closed field k of characteristic $p > 0$. Unless specified otherwise, all schemes, morphisms, and rational maps will be defined over k.

A *generalized Zariski surface* is a surface S that admits a dominant, purely inseparable, rational map $f: \mathrm{P}^2 \to S$. If we may take f so that the degree of f is p, then we call S a *Zariski surface*. Suppose the degree of f is p^e; one easily sees that the field $k(S)$ contains the field $k(x^{p^e}, y^{p^e})$, where we identify $k(\mathrm{P}^2)$ with $k(x, y)$. Thus, if there exists a map f as above, then there is also a dominant, purely inseparable, rational map $g: S \to \mathrm{P}^2$ of degree p^e. Taking p^eth roots, we see that the converse is also true. We will find this dual viewpoint useful in the sequel.

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