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# CLOSED GEODESICS AND THE FUNDAMENTAL GROUP

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In the present note we examine the existence of closed geodesics on a compact Riemannian manifold $(M, g)$ with nontrivial fundamental group. Denote the conjugacy class of an element $a \in \pi_1(M)$ by $[a]$. We define $N_g(t)$ (resp. $N_g^a(t)$) to be the number of geometrically different closed geodesics of $g$ of length $\leqslant t$ (resp. the number of those of length $\leqslant t$ in the free homotopy class defined by $[a^n]$, $n \in Z - \{0\}$). Here two closed geodesics are called geometrically different if their images are different.

THEOREM A. *Suppose that $a$ is a nontrivial element of $\pi_1(M)$ which satisfies $[a^n] = [a^m]$ for some integers $n \neq m$. Then in the space of $C^k$ metrics, $k \geqslant 4$ (including $k = \infty$), endowed with the $C^k$ topology, there exists a generic set $\mathcal{O}$ such that for any $g \in \mathcal{O}$ there exists a constant $c > 0$ such that*

$$N_g^a(t) \geqslant c \frac{t}{\ln t}$$

*for $t$ sufficiently large. In particular, there exist infinitely many closed geodesics on $(M, g)$.*

The hypothesis of the theorem is clearly satisfied if $a$ is nontrivial and has finite order. Hence we get the following immediate corollary:

COROLLARY. *Suppose that $\pi_1(M)$ is finite and nontrivial. Then for a generic metric $g$ on $M$ there exists a constant $c > 0$ such that*

$$N_g(t) \geqslant c \frac{t}{\ln t}$$

*for $t$ sufficiently large.* □

*Remark.* The hypothesis of the theorem can also be satisfied if $\pi_1(M)$ has no torsion: If $a$, $b$ are the generators of the fundamental group of the Klein bottle with the relation $bab^{-1} = a^{-1}$, then $[a] = [a^{-1}]$.

*Proof of Theorem A.* For each closed geodesic $c : R/Z \to M$ one has the linearized Poincaré map $P : E \oplus E \to E \oplus E$, where $E$ is the orthogonal complement of $\dot{c}(0)$ in $M_{c(0)}$. $P$ is given by $(x, y) \mapsto (J(1), \nabla J(1))$, where $J$ is the Jacobi field determined by $J(0) = x$, $\nabla J(0) = y$. $c$ is called hyperbolic if $P$ has no eigenvalue on the unit circle.

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