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THE CRITICAL VALUES OF ZETA FUNCTIONS ASSOCIATED TO THE SYMPLECTIC GROUP

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§1. Introduction. The purpose of this paper is to establish a formula for the projection operator on the space of C^∞ Siegel modular forms, and to apply it to the study of zeta functions of Rankin type.

Let F be a C^∞ form of weight k (precise definitions will be provided in section 2). Then there is, associated to F, a unique holomorphic cusp form $P(F)$ of weight k, satisfying the following identity of Petersson inner products:

$$\langle F, g\rangle = \langle P(F), g\rangle,$$

for all holomorphic cusp forms g of weight k. In section 2, we state a formula which relates the nth Fourier coefficient of $P(F)$ to the nth Fourier coefficient of F. The formula will be obtained in section 3 by integrating F against a certain (Bergmann-type) kernel, whose analytic properties have been studied by Godement [4].

Let $f(z)$ and $g(z)$ be holomorphic modular forms of weights k and l (on congruence subgroups of $\mathrm{Sp}(n, \mathbf{Z})$) with Fourier expansions

$$f(z) = \sum_{s \in B} a(s) e(\sigma(sz)), \qquad g(z) = \sum_{s \in B} b(s) e(\sigma(sz)).$$

Here B is the set of all positive definite n by n semi-integral matrices, z is a point on $\mathfrak{H}_n$, the Siegel upper half plane, and σ is the trace function. We define an equivalence relation on B by $s_1 \sim s_2$ if and only if $s_1 = {}^t u s_2 u$ for some $u \in \mathrm{SL}(n, Z)$. Then we put, for $\xi \in \mathbf{C}$,

$$D(\xi, f, g) = \sum_{b/\sim} a(s)\overline{b(s)} \det(s)^{-\xi},$$

where $B/\sim$ is the orbit space. This function has been studied by Shimura [8] in the one dimensional case, and by Adrianov and Kalinin [1, 2, 3], in the case in which $g = \theta$ is a certain theta series of weight $n/2$. Our first result on the critical values of $D(\xi, f, g)$ is stated in section 6, proposition 7. We define, for each fixed g of level N, a set of integers (or half integers) $\Omega(g)$, and we exhibit, for each $m \in \Omega(g)$, a holomorphic cusp form $K(m, g)$, of weight k, satsifying the following properties: For every $m \in \Omega(g)$, and every cusp form f of weight k and level N,

$$D(m, f, g) = p\langle f, K(m, g)\rangle,$$

$$K(m, g)^\sigma = K(m, g^\sigma) \qquad \text{for all } \sigma \in \mathrm{Aut}\,(\mathbf{C}),$$

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