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SOME RESULTS ON STABLE HARMONIC MAPS

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I. Introduction. A harmonic map is a differential map between two Riemannian manifolds which is a critical point of the energy integral. As a variational problem, it is natural to study so-called stable harmonic maps which have nonnegative second variation.

Let $\phi : M \to N$ be a harmonic map. If N has nonpositive sectional curvature, then by the second variational formula it is easy to see that any harmonic map is stable. Furthermore, when $M = S^n$, ϕ must be constant.

Eells and Sampson [3] showed that any differential map $\phi : S^n \to S^n$ with degree $k \neq 0$ does not have minimizing energy for $n > 2$. In particular, R. T. Smith [6] computed that the index of the identity map on S^n is $n + 1$ when $n > 2$. Certainly it is unstable.

In contrast with these, we prove a more general theorem: there is no nonconstant stable harmonic map from S^n with $n > 2$ to any Riemannian manifold (Theorem 3.1). In a certain sense it is rather like J. Simons' theorem in the case of Yang-Mills field [1].

Using a similar technique, Theorem 3.1 can be generalized to the case of the product of Euclidean spheres $S^n \times S^m$.

When $n = 2$, Eells-Wood's Theorem [4] shows that any harmonic map from S^2 to a compact Riemann surface must be holomorphic or antiholomorphic, and hence it is stable. In the case of high dimensional image manifold, there seems to be a different situation. Eells and Wood [5] pointed out there are compact simply-connected Kähler manifolds N for which any holomorphic map $\phi : M \to N$ is constant, where M is any compact Riemann surface.

II. Notations and basic formulas. We shall follow the notion of the paper [2].

Let M and N be Riemannian manifolds with dimensions m and n respectively. Throughout this paper we shall assume M is compact without boundary.

Any differentiable map $\phi : M \to N$ induces a map $\phi_* : TM \to TN$, where TM and TN are tangent bundles of M and N respectively. The induced vector bundle $E = \phi^{-1}TN \to M$ inherits a Riemannian connection from canonical connection in N as follows. Let ∇ and $\overline{\nabla}$ be the canonical connection in M and N, respectively. For any $X \in TM$ and $S \in \Gamma(E)$, the induced Riemannian connection in E is defined by

$$\tilde{\nabla}_X S \equiv \overline{\nabla}_{\phi_* X} S. \tag{2.1}$$

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