Vol. 47, No. 3 DUKE MATHEMATICAL JOURNAL © September 1980

REMARKS ON CONVERGENCE OF THE FEYNMAN PATH INTEGRALS

DAISUKE FUJIWARA

Introduction. In this note, we prove that the Feynman path integral converges, in a very strong topology, to the fundamental solution for the Schrödinger equation if the potential belongs to a class of C^{∞} functions which may be unbounded at the infinity. This class of functions contains, for example, the potential for harmonic oscillators and smooth long range potentials. To prove the convergence of the Feynman path integral, we examine the behaviour of the classical action in detail and apply a theory of oscillatory integral transformations to the integral transformation (see (4) below) introduced by Feynman [15]. Perturbation methods are not used.

There are several ways to give rigorous meaning to Feynman's path integral. Analytic continuation from the integral with respect to the Wiener measure is successful if the potential is locally singular. In this respect, we refer to, for instance, Gelfand-Yaglom [23], Cameron [9], Babitt [6] and Nelson [33]. However, the topology of the convergence seems rather weak in these cases.

One can also use the product formula of Trotter-Kato-Chernoff (cf. Chernoff [13] and Kato [25]) to give sense to Feynman's path integrals. See Nelson [33] and page 6 of Simon [36].

Itô [24] formulated Feynman's path integral as an improper integral in the Hilbert manifold of paths. This converges if the potential is the sum of a quadratic form of the position vector and the Fourier transform of a complex measure of bounded variation. Albeverio-Høegh Krohn [1] discussed this method further. In [2], they succeeded in discussing quasi-classical limit of the solution of the Schrödinger equation under additional assumptions, which imply real analyticity of the potential. Truman [37] also discussed quasi-classical limit of the solution of the Schrödinger equation under less restrictive assumptions. One can find many other interesting topics in [3].

If the Lagrangian function is infinitely differentiable, one can discuss Feynman's original formulation in [15] and [16] of the path integral and prove its convergence in a much stronger topology. In the previous papers [17], [18], [19] and [20], we discussed L^2-theory of it. We proved that Feynman's approximating sequence converges, in the uniform operator topology, to the fundamental solution of the Schrödinger equation. Using the result of [20], Yajima [38] discussed quasi-classical limit of the quantum scattering theory.

In the present note, we shall prove that the *integral kernel* of Feynman's

Received September 6, 1979. Revision received April 18, 1980.