Vol. 47, No. 1 DUKE MATHEMATICAL JOURNAL © March 1980

BOUNDARY VALUES AND ESTIMATES FOR HOLOMORPHIC FUNCTIONS OF SEVERAL COMPLEX VARIABLES

BY STEVEN G. KRANTZ

Let $\mathcal{D} \subset\subset \mathbf{C}^n$ be an open domain with smooth boundary. If $f \in C(\overline{\mathcal{D}})$ is holomorphic on $\mathcal{D}$ and $f|_{\partial\mathcal{D}}$ is smooth, then f has corresponding regularity on $\overline{\mathcal{D}}$. This is a very special instance of the regularity theory for elliptic boundary value problems. More subtle phenomena occur if one specializes to $\mathcal{D} \subset\subset \mathbf{C}^n$, $n > 1$. In many instances, weaker hypotheses imply stronger conclusions. Very fine results may be obtained if one takes into account the differential geometry of $\partial\mathcal{D}$. The purpose of this paper is to investigate these phenomena. The results contained herein are dual to those in [11].

Section 1 collects the requisite notations. In Section 2, we recall some known results and formulate the first main result of the paper: that if $\mathcal{D} \subset\subset \mathbf{C}^n$ is smoothly bounded then there is a tangent vector field η on $\partial\mathcal{D}$ so that if f is holomorphic on $\mathcal{D}$ and the radial boundary values of f on $\partial\mathcal{D}$ are smooth (in an appropriate sense) along integral curves of η, then f is smooth on $\overline{\mathcal{D}}$. Section 3 contains the proof of this theorem, and in Section 4 we isolate an allied result which is of independent interest. Section 5 contains a differential geometric refinement of the first theorem and Section 6 contains some further results. Section 7 provides an example to illustrate the results.

I am very grateful to Robert Jensen and to Michael Taylor for several helpful discussions about Theorem 2.4. Indeed, they each provided me with a proof of Theorem 4.1 which provides an alternate approach to Theorem 2.4 and is of considerable independent interest. I am also grateful to N. Th. Varopoulos for a helpful discussion of Theorem 2.4 in the strongly pseudoconvex case.

1. Definitions and notation. Throughout, we will assume that $\mathcal{D} \subset\subset \mathbf{C}^n$ is a connected open set with smooth (C^∞) boundary. For convenience, we will operate in the standard Euclidean metric on $\mathbf{C}^n$ (although any non-degenerate Hermitian metric will do). If $P \in \partial\mathcal{D}$, we let ν_P denote the unit outward normal to $\partial\mathcal{D}$ at P. If $z \in \mathbf{C}^n$, we let

$$\delta(z) = \delta_{\mathcal{D}}(z) = \text{distance of } z \text{ to } \partial\mathcal{D}.$$

In what follows, the smoothness hypotheses on $\partial\mathcal{D}$ can be considerably weakened but the exposition is clearer if we finesse this point.

Received March 1, 1979. Revision received July 16, 1979. This work was partially supported by NSF Grant No. MCS-77-02213. The author is grateful to the Université de Paris–Sud for its hospitality during a part of this work.