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REGULARLY RELATED LATTICES IN LIE GROUPS

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Let G be a connected Lie group and let H and K be two lattices in G, that is, discrete subgroups such that G/H and G/K have finite G-invariant measures. Let K act on G/H by restricting the natural action of G on G/H. We consider the question of when the orbit structure for the action of K on G/H is nicely behaved. More than a dozen equivalent conditions for the nice behavior of orbit structure for transformation groups are discussed in papers by Glimm [4] and Effros [3]. These conditions are all stated globally. In this paper we consider local versions of three of the most convenient of these conditions—one a topological condition and the other two measure theoretical—and show that even local nice behavior can only occur under quite special circumstances. Undoubtedly versions of many of the other conditions of Glimm and Effros could be used to reach the same conclusion equally well.

MAIN THEOREM. *Let G be a connected Lie group, and let H and K be lattices in G. Suppose that any one of the following conditions holds:*

1. *The set of points of G/H whose K-orbits are open in their closure,* i.e., *relatively discrete, (which is a measurable set) has strictly positive measure.*
2. *There is a K-invariant measurable subset, F, of strictly positive measure in G/H such that the quotient Borel structure in $K\backslash F$ is countably separated.*
3. *There is a measurable subset of G/H of strictly positive measure which meets each K-orbit in at most one point.*

Then $G \cong \mathbb{R}^n \times C$ for some compact group C. Furthermore, $H \cap K$ contains a subgroup which is central in G and of finite index in both H and K. In particular, every K-orbit in G/H is finite, (as will be every H-orbit in G/K).

As will be explained in detail later, condition 2 is a weak version of Mackey's condition [7, 8] that H and K be regularly related, which plays an important role in his theory of induced representations. Thus if H and K are regularly related, one can draw the same conclusions.

I was led to consider the question of nice orbit structures by my study [10, 12] of the von Neumann algebra obtained from the action of K on G/H, which will be a finite von Neumann algebra in the case at hand. The orbit structure

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