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THE SPECIAL VALUES OF THE ZETA FUNCTIONS ASSOCIATED WITH HILBERT MODULAR FORMS

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The zeta functions to be studied are of the following two types:

$$D(s, \mathbf{f}, \mathbf{g}) = \sum C(\mathfrak{n}, \mathbf{f})C(\mathfrak{n}, \mathbf{g})N(\mathfrak{n})^{-s}, \tag{I}$$

$$D(s, \mathbf{f}, \omega) = \sum C(\mathfrak{n}, \mathbf{f})\omega(\mathfrak{n})N(\mathfrak{n})^{-s}. \tag{II}$$

Here $\mathbf{f}$ and $\mathbf{g}$ are Hilbert modular forms defined relative to a totally real algebraic number field F; ω is a Hecke (ideal) character of F of finite order; $\mathfrak{n}$ runs over all integral ideals of F; $C(\mathfrak{n}, \mathbf{f})$ and $C(\mathfrak{n}, \mathbf{g})$ are the "Fourier coefficients" of $\mathbf{f}$ and $\mathbf{g}$. The theorems in Section 4 will assert certain algebraicity properties of the values of $D(s, \mathbf{f}, \mathbf{g})$ and $D(s, \mathbf{f}, \omega)$ at an integer or a half-integer, when $\mathbf{f}$ is a primitive cusp form. This generalizes our previous results in the elliptic modular case ([15], [17]). Our methods of proof are the same as in that case. In particular, it is indispensable to investigate the behavior of a modular form under the action of automorphisms on its Fourier coefficients.

If F has degree n, Hilbert modular forms are defined with respect to a factor of automorphy of the form

$$\prod_{\nu=1}^{n} (c_\nu z_\nu + d_\nu)^{k_\nu},$$

where $k_1, \cdots, k_n$ are positive integers. The modular forms with arbitrary $k_1, \cdots, k_n$ are as natural and important as those in the special case $k_1 = \cdots = k_n$. In fact such "multiple weights" present many interesting features which do not exist in the one-dimensional case. This is one of the reasons why we have taken up the present work. Another motive is that there is an application of our results to the theory of arithmetic automorphic forms, which the author hopes to discuss on some future occasion.

One final remark may be added. The series of type (I) are significant on their own merits; they should not be regarded as auxiliary objects to the study of the series of type (II), as the introduction of the previous paper [15] might have misleadingly suggested.

Notation. If R is an associative ring with identity element, $R^\times$ denotes the group of all invertible elements of R. A diagonal matrix with diagonal entries $c_1, \cdots, c_n$ is denoted by diag $[c_1, \cdots, c_n]$. We put $e(z) = e^{2\pi iz}$ for $z \in \mathbf{C}$ and

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