Vol. 44, No. 4 DUKE MATHEMATICAL JOURNAL© December 1977

ANALYTIC SUBGROUPS OF AFFINE ALGEBRAIC GROUPS

ANDY R. MAGID

A connected linear algebraic group over $\mathbb{C}$ can be regarded as a complex analytic group. This paper studies integral analytic subgroups in the sense of [2, Defn. 1, p. 306] of such an algebraic group which are dense in the Zariski topology of the algebraic group. Any connected complex analytic group which has a faithful complex analytic representation can be regarded as such an integral subgroup of the Zariski closure in a general linear group of the image of a faithful representation, and our study yields information about such groups.

Let G be a connected linear algebraic group and H a Zariski-dense integral analytic subgroup. We show that there is an algebraic torus T in G such that $G = HT$, and such that the Lie algebra of G is the direct sum of the Lie algebras of H and T. Such a T is called a complementary torus to H in G. Under suitable conditions on H and G we can guarantee that $H \cap T$ is discrete, or even finite. An important special case occurs when $H \cap T = \{e\}$. We show that in this case the affine coordinate ring of G induces a left algebraic group structure on H in the sense of [7, Defn. 2.1]. Conversely, all left algebraic group structures on H are shown to arise in this fashion. We further show that H is a semi-direct product KQ where K is a simply connected solvable normal subgroup of H and Q is a reductive algebraic subgroup of G, and that there is a complementary torus to H in G centralized by Q. This provides a proof that a faithfully representable analytic group has a nucleus, which is a result of [5, p. 113], avoiding the use of compact real subgroups.

Also, for lack of a suitable reference we include as an appendix a proof of the theorem that a faithfully representable finite cover of an algebraic group is again an algebraic group.

All our algebraic groups are over $\mathbb{C}$, and we use $k[G]$ to denote the affine coordinate ring of the algebraic group G. $L(H)$ denotes the Lie algebra of the analytic group H. We call an analytic group of the form $(\mathbb{C}^*)^{(n)}$ a *multiplicative torus*. A homomorphism from an analytic group to $\mathbb{C}$ is called an *additive character*. We recall the following elementary facts: a multiplicative torus can be regarded as an algebraic group, and any analytic homomorphism between two such tori is algebraic. Moreover, a multiplicative torus has no non-trivial additive characters. We also need the following:

Received August 12, 1977. Revision received September 6, 1977. Partially supported by National Science Foundation.