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A FAMILY OF SMOOTH HYPERBOLIC HYPERSURFACES IN $\mathbb{P}_3$

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We construct a family of smooth hypersurfaces V_ϵ of high degree in $\mathbb{P}_3$ parametrized by the disc so that V_ϵ is hyperbolic (in the sense of Kobayashi, see [3]) for $\epsilon \neq 0$, but V_0 is not. This resolves two open questions posed in [4]. First, by the Lefschetz theorem, the V_ϵ are simply-connected compact hyperbolic surfaces. Second, we have a family of diffeomorphic surfaces all but one of which are hyperbolic. This is in contrast to the situation for compact Riemann surfaces, where hyperbolicity depends only on topological information.

Our family of hypersurfaces in $\mathbb{P}_3$ is given by

$$V_\epsilon = \{z_0^d + z_1^d + z_2^d + z_3^d + (\epsilon z_0 z_1)^{d/2} + (\epsilon z_0 z_2)^{d/2} = 0\}$$

where d is even and ≥ 50. Since V_0 is the Fermat surface, which is nonsingular, we know the nearby V_ϵ are nonsingular. The variety V_0 contains complex lines for all d, for example $z_1 = \mu z_0$, $z_3 = \eta z_2$ where $\mu^d = \eta^d = -1$.

The task of showing the V_ϵ are hyperbolic for $\delta \neq 0$ was greatly simplified by a theorem of the first author [1] which affirms that a compact complex manifold is hyperbolic provided that it admits no non-constant holomorphic maps from $\mathbb{C}$. Modulo a small amount of combinatorial manipulation, we can now apply a result of the second author [2] about holomorphic maps from $\mathbb{C}$ to Fermat varieties. The case of the theorem we need is the following:

THEOREM. *Any holomorphic map* $\mathbb{C} \xrightarrow{(g_0, \cdots, g_n)} W_d \subset \mathbb{P}_n$, *where* $W_d = \{w_0^d + w_1^d + \cdots + w_n^d = 0\}$ *has image contained in a linear subspace of dimension* $\leq \dfrac{n-1}{2}$, *provided* $d \geq n^2$.

Further, there is a partition $I_1, \cdots, I_k$ *of the indices* $\{0, \cdots, n\}$ *so* g_i/g_j *is constant if* i, j *are in the same* I_q. *Each* I_q *has* ≥ 2 *elements.*

This is related to our situation by setting

$$\begin{aligned} g_0 &= f_0^2 & g_3 &= f_3^2 \\ g_1 &= f_1^2 & g_4 &= \epsilon f_0 f_1 \\ g_2 &= f_2^2 & g_5 &= \epsilon f_0 f_2 \end{aligned}$$

Then $\mathbb{C} \xrightarrow{(g_0, \cdots, g_5)} W_{d/2} \subset \mathbb{P}_5$ if $\mathbb{C} \xrightarrow{(f_0, \cdots, f_3)} V_\epsilon \subset \mathbb{P}_3$. We now have:

Case 1. Some $f_i \equiv 0$.

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