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INTEGER POINTS ON A SPECIAL CUBIC SURFACE

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1. Introduction

Elementary algebraic geometry can play an important role in determining rational points on projective varieties, but it may also be used in certain instances to find an infinity of integer points on the variety V under consideration. The easiest way of doing this is to exhibit a parametric polynomial solution of the equation of V.

The most well known example is when V is the cubic surface

$$x^3 + y^3 + z^3 = 1;$$

Euler showed that V possesses an infinity of integer solutions by means of the identity

$$(-9t^4 - 3t)^3 + (9t^4)^3 + (9t^3 + 1)^3 = 1.$$

We proceed to determine such parametric solutions of the equation

$$x^3 + y^3 + z^3 = x + y + z. \tag{1}$$

This appears to have been first mentioned in 1915, when Bökle [1] proposed the problem of showing that the equivalent equation

$$\binom{x}{3} + \binom{y}{3} = \binom{z}{3}$$

has infinitely many integer solutions. A proof appeared in the same journal four years later (Rohr [1]), and was essentially rediscovered by Sierpiński [1]. Wunderlich [1] listed a large number of small integral solutions, and Edgar [1], Oppenheim [1] gave different constructions from that of Sierpiński for producing infinite classes of solutions. The first parametric solutions are given by Oppenheim [2], who exhibits explicit polynomials of degrees 4 and 7, and describes how to construct infinitely many solutions using properties of the hyperbolic functions.

2. The geometry of the surface

All the requisite geometry of this section is succinctly described in Swinnerton-Dyer [1].

The cubic equation (1) corresponds to the cubic surface in projective three space given by

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