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ON THE SIMULTANEOUS APPROXIMATION OF CERTAIN NUMBERS

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1. Introduction. Let $f(x_1, \cdots, x_n)$ be a function such that $f(\alpha_1, \cdots, \alpha_n)$ is known to be transcendental for all n-tuples $(\alpha_1, \cdots, \alpha_n)$ of algebraic numbers (except perhaps for certain specified tuples).

The following problem frequently arises in the literature: Prove that $f(\alpha_1, \cdots, \alpha_n)$ is transcendental, even if $\alpha_1, \cdots, \alpha_n$ are no longer algebraic numbers, but numbers which can be approximated very well by algebraic numbers. Sometimes one can even give a measure for the approximability of thus constructed transcendental numbers $f(\alpha_1, \cdots, \alpha_n)$. In that case, one can use the following, more symmetrical, formulation of the same problem: Prove that the $n + 1$ numbers $\alpha_1, \cdots, \alpha_n$ and $f(\alpha_1, \cdots, \alpha_n)$ cannot simultaneously be approximated very well by algebraic numbers.

In the case of $f(x_1, x_2) = x_1^{x_2}$, thus in the case of α, β and α^β, say, there are some early results for which the reader is referred to [7, p. 103]. Some later developments for this and related cases can be found in [1], [8] and [9].

In this paper, we consider the case of $f(x_1) = e^{x_1}$, i.e., we consider simultaneous approximations of α and e^α by algebraic numbers. Our main result will be formulated in terms of the heights H_1 and H_2 of the algebraic numbers approximating α and e^α, respectively. We already remark that this estimate is as sharp as possible for each of these heights apart; the reader can easily verify this for algebraic α and for algebraic e^α (compare e.g. [10, formula (4) or (4′)]). Further, another variant of this result will be given, in which $H = \max(H_1, H_2)$ is used instead of H_1 and H_2. This second estimate is far from best possible, but when H_1 and H_2 have approximately the same order of magnitude, it appears to be somewhat sharper than the first one.

A second purpose of the present paper is to use an auxiliary result which can be found in [3]. This is done mainly to illustrate its usefulness and to indicate in what way it can be used.

The proofs in this paper use Gel'fond's method, with some refinements related to those introduced by Fel'dman in e.g. [5].

2. Formulation of the results.

THEOREM 1. *Let α be a nonzero complex number and let N_1 and N_2 be positive integers.*

There exists an effectively computable, positive number C, depending only on

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