AN APPLICATION OF SECOND VARIATION TO SUBMANIFOLD THEORY

JOHN DOUGLAS MOORE

1. Introduction. In this note we will use second variation to give a proof of the following theorem:

THEOREM. *Let $\bar{M}$ be a complete simply connected Riemannian manifold whose sectional curvatures $\bar{K}(\sigma)$ satisfy the inequalities*

$$a \leq \bar{K}(\sigma) \leq b \leq 0, \tag{1}$$

M a compact Riemannian manifold whose sectional curvatures $K(\sigma)$ satisfy $K(\sigma) \leq a - b$. If $\dim \bar{M} < 2 \dim M$, *then M possesses no isometric immersion in $\bar{M}$.*

This theorem generalizes theorems of O'Neill [4] and Stiel [6]. Gray has proven the above theorem under the additional assumption that $\bar{M}$ be a symmetric space [1, Corollary 3.5]. Using the method described here, it is also possible to make an extension of Theorem 3.4 of [1], but we will not enter into the details. The central idea of the proof consists of comparing the index form of $\bar{M}$ with the index form of a constant curvature manifold.

Our thanks go to George Kelley for critically reading this note.

2. Proof of the Theorem. We assume the existence of an isometric immersion $f: M \to \bar{M}$ which satisfies the hypotheses of the theorem, and derive a contradiction.

Recall standard constructions regarding second variation [3]. Choose a point $\bar{p} \in \bar{M}$. Let Ω be the set of pairs (q, γ), where $q \in M$ and $\gamma: [0, 1] \to \bar{M}$ is a smooth path satisfying $\gamma(0) = \bar{p}$, $\gamma(1) = f(q)$. (To avoid clumsy notation, we will write γ for the pair (q, γ), and identify $q \in M$ with its image $f(q)$). We have a function

$$J: \Omega \to \mathbf{R} \quad \text{defined by} \quad J(\gamma) = \frac{1}{2} \int_0^1 \langle \gamma'(t), \gamma'(t) \rangle \, dt,$$

where $\langle\ ,\ \rangle$ is the Riemannian metric of $\bar{M}$. A critical point for J is a constant speed geodesic which hits M orthogonally. If $\gamma \in \Omega$, let $T_\gamma\Omega$ denote the space of smooth vector fields X along γ such that $X(0) = 0$ and $X(1)$ is tangent to M. At a critical point γ, we can define the Hessian d^2J on $T_\gamma\Omega$; a calculation shows that

$$d^2J(X, Y) = I(X, Y) + \langle \alpha(X(1), Y(1)), \gamma'(1) \rangle,$$

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