ON COMMUTANTS OF REDUCTIVE OPERATOR ALGEBRAS

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1. Introduction. An algebra of bounded linear operators on a Hilbert space is *reductive* if it is weakly closed, contains the identity, and has the property that each of its invariant subspaces is reducing. The *reductive algebra problem* is the question: is every reductive algebra self-adjoint? The known partial results on this problem are discussed in [9].

We consider the following properties an operator T may have:

P.1. If every invariant subspace of T is reducing, then T is normal.

P.2. If $\mathfrak{A}$ is a reductive algebra containing T, and $T \in \mathfrak{A}'$ ($\mathfrak{A}'$ denotes the commutant of $\mathfrak{A}$), then $T^* \in \mathfrak{A}'$.

P.3. If $\mathfrak{A}$ is any reductive algebra and $T \in \mathfrak{A}'$, then $T^* \in \mathfrak{A}'$.

Obviously an affirmative answer to the reductive algebra problem implies P.3 for all T. Also P.3 implies P.2, and P.2 implies P.1. It is known that P.1 holds for many classes of operators, and Dyer and Porcelli ([3], [4]) have shown that the truth of P.1 for all T is equivalent to existence of non-trivial invariant sub-spaces for all T. Azoff, Fong and Gilfeather [2] have modified the techniques of Dyer and Porcelli to prove that the truth of P.2 for all T is equivalent to existence of non-trivial hyperinvariant subspaces for all non-scalar T. Also, P.3 is known to hold for algebraic T [9, Lemma 9.3] and for operators similar to normal operators [9, Lemma 9.14].

In this note we make some observations concerning P.2 and prove that P.3 holds for n-normal operators and for compact operators. This last result is shown to imply P.1 for operators quasi-similar to compact operators.

2. Some observations concerning P. 2.

THEOREM 1. *If T is compact, then T has property* P.2.

Proof. Let $\mathfrak{M}$ denote the nullspace of T; clearly $\mathfrak{M}$ reduces the given reductive algebra $\mathfrak{A}$. Now $\mathfrak{A}|\mathfrak{M}^\perp$ has all its invariant subspaces reducing and contains the injective compact operator $T|\mathfrak{M}^\perp$. Thus, by [11] or [1], the weak closure of $\mathfrak{A}|\mathfrak{M}^\perp$ is self-adjoint. Hence $(T|\mathfrak{M}^\perp)^*$ commutes with $\mathfrak{A}|\mathfrak{M}^\perp$ and, since $T^*|\mathfrak{M} = 0$, it follows that $T^* \in \mathfrak{A}'$.

Theorem 1 is strengthened in Theorem 4 below.

THEOREM 2. *The following classes of operators have property* P.2:

(a) *spectral operators with compact quasi-nilpotent parts.*

(b) *operators T such that $1 - T^*T$ or $T - T^*$ are in some Schatten $\mathcal{C}_p$ class.*

Received February 1, 1974. This work was partially supported by the Canada Council.