ROSENTHAL SETS AND RIESZ SETS

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In this paper $\mathbf{T}$ is the circle group and $\mathbf{Z}$ the additive group of integers. Denote by $M(\mathbf{T})$ the usual convolution algebra of Borel measures on $\mathbf{T}$. The nth Fourier–Stieltjes coefficient $\hat{\mu}(n)$ of a measure $\mu \in M(\mathbf{T})$ is defined by

$$\hat{\mu}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\theta}\, d\mu(\theta) \qquad (n \in \mathbf{Z}).$$

Let $L^p(\mathbf{T})$ denote the usual Lebesgue space of index $1 \leq p \leq \infty$ and $C(\mathbf{T})$ those $f \in L^\infty(\mathbf{T})$ which are continuous. For any subset E of $\mathbf{Z}$ and subspace $B(\mathbf{T})$ of $L^1(\mathbf{T})$ put

$$B_E(\mathbf{T}) = \{f \in B(\mathbf{T}) : \hat{f} = 0 \quad \text{off} \quad E\}.$$

A subset $E \subset \mathbf{Z}$ is called a Sidon set if whenever $f \in L^\infty(\mathbf{T})$ and supp $\hat{f} \subset E$ then $\sum |\hat{f}(n)| < \infty$. Thus for E a Sidon set $C_E(\mathbf{T}) = L_E^\infty(\mathbf{T})$. By definition a subset E for which

$$C_E(\mathbf{T}) = L_E^\infty(\mathbf{T}) \tag{1}$$

is called a Rosenthal set.

Examples of non Sidon sets E satisfying (1) were first given by Rosenthal in [9]. More precisely the following statement was proved: Let $E_n = \{1, 2, \cdots, n\}$. Then $E^1 = \bigcup_{n=0}^\infty (19)^n n! E_{n+1}$ and $E^2 = \bigcup_{n=1}^\infty (2n)! E_{2n}$ are Rosenthal sets. Furthermore, E^1 and E^2 are not Sidon sets. Subsequently Blei in [2] proved that any non Sidon set E has a subset which is Rosenthal but not Sidon.

Our purpose in this paper is to give several extensions of the F. and M. Riesz theorem via Rosenthal sets. The classical F. and M. Riesz theorem can be stated as:

THEOREM 1. *Suppose $\mu \in M(\mathbf{T})$ and $\hat{\mu}(n) = 0$ for all $n > 0$. Then μ is absolutely continuous with respect to Lebesgue measure on* $\mathbf{T}$.

We prove this extension of Theorem 1:

THEOREM 2. *Let E be a Rosenthal set. Let $\mu \in M(\mathbf{T})$ and suppose $\hat{\mu}(n) = 0$ for all positive n such that $n \notin E$. Then μ is absolutely continuous with respect to Lebesgue measure on* $\mathbf{T}$.

Proof. We cut off $\hat{\mu}$ as follows: The Hardy space $H^\infty(\mathbf{T})$ consists precisely of those $f \in L^\infty(\mathbf{T})$ for which the Fourier coefficients $\hat{f}(n) = 0$ for all $n < 0$. Well, let $f \in H^\infty(\mathbf{T})$. Then

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