DISTORTION OF HYPERBOLIC AREA UNDER QUASICONFORMAL MAPPINGS

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1. Introduction. In this paper we show that the techniques of [4] and [5], in which the effect of quasiconformal mappings on euclidean area is investigated, can be adapted to prove a similar result for hyperbolic area. Indeed, it follows as an easy corollary of [4; Theorem 1], quoted as Theorem A in Section 2, that for each measurable subset of the unit disk

$$m_h(f(E)) \leq C(R_h\,,\, K)m_h(E)^{K^{-a}},$$

where f is a K-quasiconformal self-mapping of the unit disk, m_h is hyperbolic area, C is a constant depending on K and R_h, the hyperbolic radius of the set E, and a is a certain universal constant, $1 \leq a \leq 20$. (It was reported in [5] that $1 \leq a \leq 17$, but a computational error there results in the correct estimate of 20, using the same techniques.) This estimate however is imprecise in its behavior as $K \to 1$ and, indeed, is misleading in its behavior as $m_h(E) \to \infty$. It appears that $m_h(f(E))$ behaves no worse than a constant times $m_h(E)^{K^{-a}}$, whereas in fact it can become as big as $K\, m_h(E)$ (cf. [3; Theorem 3] reproduced as Theorem B in Section 2). The difficulty is that as $m_h(E) \to \infty$ so does its hyperbolic radius R_h and, subsequently, $C(R_h\,,\, K)$. The problem then is to find a distortion function $\psi(\tau)$, $0 < \tau < \infty$, such that $m_h(f(E)) \leq \psi(m_h(E))$ and which behaves correctly for τ near 0 and ∞ and as $K \to 1$. The "correct" behavior is something that must be decided. The distortion function will of course depend on K. It is somewhat surprising that it must also depend on R_h, the hyperbolic radius of the set E. In [3; Theorem 3] (Theorem B), where the distortion of *disks* is investigated, a sharp estimate is found which is independent of R_h. Since this estimate is sharp, the behavior of this distortion function near 0 and ∞ is taken as the "correct" behavior in the general case. That dependency on R_h is essential in the general case is demonstrated by an example at the end of Section 2.

2. Preliminaries. Suppose D is a simply connected domain of hyperbolic type in the finite $z = x + iy$ plane, Ω, and that g is a conformal mapping of D onto the unit disk $U : |z| < 1$. Then the function $|g'|/(1 - |g|^2)$ is independent of the choice of g, and it defines the hyperbolic metric in D. In particular the *hyperbolic area* of a measurable set $E \subset D$ is

$$m_h(E) = \iint_E \frac{|g'(z)|^2\, d\sigma}{(1 - |g(z)|^2)^2}. \tag{1}$$

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