REPRESENTABILITY OF BINARY QUADRATIC FORMS OVER A BÈZOUT DOMAIN

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1. Introduction. By a form we shall mean a binary quadratic form in indeterminates X and Y with coefficients in a Bézout domain R, that is, an integral domain in which every finitely-generated ideal is principal. Such a form $lX^2 + mXY + nY^2$ will be called primitive if $(l, m, n) = R$. Δ will denote a nonsquare element of R which is the discriminant of some binary quadratic form in R. If the characteristic of R is 2, no such Δ exists; so we assume throughout that char $(R) \neq 2$.

If $f(X, Y) = aX^2 + bXY + cY^2$ is a given form and $g(X, Y)$ is a form of discriminant Δ, we say that $f(X, Y)$ is representable by $g(X, Y)$ if there exist elements $p, q, r, s \in R$ with $ps - qr \neq 0$ such that $f(X, Y) = g(pX + qY, rX + sY)$. If such elements p, q, r and s exist, we call (p, q, r, s) a representation of f by g. Clearly a necessary condition for the representability of f by g is

$$\begin{aligned}\text{discrim } (f(X, Y)) &= \text{discrim } (g(pX + qY, rX + sY)) \\ &= (ps - qr)^2 \text{ discrim } (g(X, Y)) \\ &= \Delta k^2,\end{aligned}$$

where k is a nonzero element of R. From now on we assume that $f(X, Y) = aX^2 + bXY + cY^2$ is a given form of discriminant Δk^2, where k is a fixed nonzero element of R, and that $g(X, Y) = lX^2 + mXY + nY^2$ denotes an arbitrary primitive form of discriminant Δ. A representation (p, q, r, s) of $f(X, Y)$ by the form $g(X, Y)$ will be called proper if $ps - qr = k$ and improper if $ps - qr = -k$.

In the classical case $R = Z$ (the domain of rational integers) for discriminants given by

$$-\Delta = 3, 4, 7, 8, 11, 19, 43, 67, 163$$

one of us [7], extending results of Mordell [4] (see also [5]) and Pall [6] (see also [8]), has determined necessary and sufficient conditions for a positive-definite form of discriminant Δk^2 to be representable by a positive-definite form of discriminant Δ, as well as the number of such representations. Later the authors of this paper extended these results to all field discriminants Δ, replacing the use of unique factorization in the ring of integers of $Q(\sqrt{\Delta})$ by a relationship between certain ideals of this ring and representations of $f(X, Y)$ by forms of discriminant Δ. In the present paper we replace the use of these ideals by

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