CONCRETE SEMISPACES AND LEXICOGRAPHIC ORDER

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A *semispace* at the vector p in a real linear space E is a maximal convex subset of $E \sim \{p\}$. Hammer introduced semispaces in 1955 and Klee determined the structure of semispaces in 1956. The present paper is a sequel to *Concrete semispaces and lexicographic separation of convex sets* in which *concrete semispaces* in locally convex spaces were introduced. Whenever S is a semispace at the origin θ in E, both S and $S' = S \cup \{\theta\}$ are cones and the linear ordering induced on E by S' is termed a *lexicographic* order. The class of subsets of a locally convex space which admit maximal elements for each lexicographic ordering of E determined by a *well-ordered* concrete semispace is of particular interest. A main result shows that this class includes the compact sets. Any closed and convex set which belongs to the above class is the closed convex hull of its lexicographic maxima and this provides a generalization of the Krein-Milman theorem. An example is given to show that not all extreme points of a compact convex set are necessarily lexicographic maxima for one of the orderings considered.

1. Preliminaries. A collection F of linear functionals on a real linear space E is *total* if for each nonzero x in E there is f in F such that $f(x) \neq 0$. In this case an anti-reflexive linear ordering r of F is *usable* provided that for each nonzero vector x there is a first member f_x in F such that $f_x(x) \neq 0$. In particular, if r well-orders F, then r is usable. Whenever r is usable we define

$$S(F, r) = \{x \in E : f_x(x) > 0\}$$

and note that $S(F, r)$ is a semispace at θ in E. Klee proved in [8] that every semispace S at θ can be represented in this manner. A representation $S(F, r)$ for S in which each f in F is the first functional f_x for some x in E is termed a *Klee representation* for S. A semispace S is *well-ordered* if S has a representation $S(F, r)$, where r is a well-ordering for F. Hereafter "semispace" will mean "semispace at θ".

A semispace S in a locally convex Hausdorff space E is said to be *concrete* provided S has a representation $S(F, r)$, where F is a subset of the topological dual E'. We will denote by $\mathcal{S}$ the well-ordered concrete semispaces at θ in E.

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