STONE LATTICES

BY RAYMOND BALBES AND ALFRED HORN

A Stone lattice is a pseudo-complemented distributive lattice L such that $x^* + x^{**} = 1$ for all $x \varepsilon L$. The problem of characterizing such lattices was raised by M. H. Stone. A first solution was given by G. Grätzer and E. T. Schmidt [4] in terms of prime ideals. Later G. Grätzer [5] proved that a lattice is a Stone lattice if and only if it is a * sublattice of the lattice of all ideals of a complete Boolean algebra.

In this paper we shall prove that a lattice is a Stone lattice if and only if it is a * sublattice of a direct product of dense distributive lattices with 0 and 1. In [4], it was shown that every finite Stone lattice is a direct product of such dense lattices, and this result was extended to Noetherian Stone lattices by J. Varlet [6]. We shall also characterize relative Stone lattices with 1 as $\rightarrow$ sublattices of direct products of linearly ordered sets with 1. In the last section, we give a description of free Stone lattices. In particular, a free Stone lattice with finitely many free generators is a direct product of free distributive lattices to which 0 and 1 are adjoined.

1. Definitions and preliminaries. If S is a subset of a lattice, $\Pi(S)$ and $\Sigma(S)$ denote the greatest lower bound and least upper bound of S. If $S = \{x, y\}$, $\Pi(S)$ is written as xy and $\Sigma(S)$ is written as $x + y$. We shall also use $\prod_{i \varepsilon I} L_i$ to denote the direct product of a family $\{L_i : i \varepsilon I\}$ of lattices. The smallest and largest elements of a lattice L are denoted by 0 (or 0_L) and 1.

Suppose L is a lattice with 0. If $x \varepsilon L$, the *pseudo-complement* x^* of x is the largest $y \varepsilon L$ such that $xy = 0$. L is *pseudo-complemented* if every member of L has a pseudo-complement. L is said to be *dense* if $x^* = 0$ for all $x \varepsilon L$, $x \neq 0$. A * sublattice of a pseudo-complemented lattice is a sublattice containing 0 and closed under *. If L and M are pseudo-complemented lattices, a map $f : L \rightarrow M$ is called a * homomorphism if f is a lattice homomorphism such that $f(0) = 0$ and $f(x^*) = f(x)^*$ for all $x \varepsilon L$.

We have defined Stone lattices. We take this opportunity to point out that the class of Stone lattices is an equational class. More generally, let us define a *semi-lattice* to be a partially ordered set such that the greatest lower bound xy of any two elements x, y exists. Then we have the following.

THEOREM 1.1. *An abstract algebra* $\langle L, \cdot, {}^*, 0 \rangle$ *is a pseudo-complemented semi-lattice if and only if it satisfies the following identities*:

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