GROUPS WITH MANY EQUAL CLASSES

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In this paper we consider finite groups G, which contain a proper normal subgroup N such that all of the conjugacy classes of G which lie outside of N have equal sizes. A nonabelian group with this property will be said to satisfy condition (*). Our main result is

THEOREM. *Let G satisfy (*) with respect to N. Then either G/N is cyclic, or else every nonidentity element of G/N has prime order. In the first situation, G has an abelian Hall π-subgroup and a normal π-complement, where π is the set of primes dividing the index, $|G : N|$.*

The first possibility occurs, for instance, when G is a Frobenius group with kernel N and a cyclic complement. It is not hard to construct examples of the second possibility where G/N is isomorphic to the symmetric group Σ_3 or to the nonabelian group of order p^3 and period p for any odd prime. These examples will be sketched at the end of the paper.

Before proving the theorem we draw some easy conclusions from the hypothesis (*) and prove some preliminary results. Let $x \in G - N$. Then $|\mathbf{C}(x)|$ is independent of x. Let $C_x = \mathbf{C}(\mathbf{C}(x))$. Since $x \in \mathbf{C}(x)$, $C_x \subseteq \mathbf{C}(x)$ so $C_x = \mathbf{Z}(\mathbf{C}(x))$ is abelian. Since G is not abelian, $C_x < G$ and $G = N \cup \bigcup_x C_x$. Now suppose for $x, y \in G - N$ that $C_x \cap C_y \not\subseteq N$. Choose $u \in C_x \cap C_y - N$. Then $\mathbf{C}(u) \supseteq \langle \mathbf{C}(x), \mathbf{C}(y) \rangle$. Since $|\mathbf{C}(u)| = |\mathbf{C}(x)| = |\mathbf{C}(y)|$, we have $\mathbf{C}(x) = \mathbf{C}(y)$ and $C_x = C_y$. This also shows that for any $u \in C_x - N$ we have $\mathbf{C}(x) = \mathbf{C}(u)$. This situation suggests the following definition which will be exploited in Proposition 4.

DEFINITION 1. Suppose that $N \triangle G$ and $G = N \cup \bigcup_i H_i$ where the $H_i < G$ are subgroups satisfying $H_i \cap H_j \subseteq N$ when $i \neq j$. In this situation we say that G is *partitioned relative to N.*

The following lemma will be used to prove Proposition 3, which is a weak version of the theorem.

LEMMA 2. *Let G satisfy (*) and let $x \in G - N$. Suppose $\bar{x} = xN \in G/N = \bar{G}$ is not a p-element of $\bar{G}$. Then an S_p subgroup of $\mathbf{C}(x)$ is central in $\mathbf{C}(x)$. If the order, $o(\bar{x})$ is divisible by two distinct primes, then $\mathbf{C}(x)$ is abelian and $\mathbf{C}(x) = C_x$.*

Proof. Suppose $y \in \mathbf{C}(x)$ is a p-element so $y^{p^r} = 1$. Then $(xy)^{p^r} = x^{p^r} \notin N$. We have $\mathbf{C}(xy) \subseteq \mathbf{C}((xy)^{p^r})$, and these groups have equal orders so

$$\mathbf{C}(xy) = \mathbf{C}((xy)^{p^r}) = \mathbf{C}(x^{p^r}) \supseteq \mathbf{C}(x).$$

Received September 16, 1968.