A SONINE TRANSFORM

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It is known that if f is a square integrable function and

$$g(x) = \int_0^x f(t) J_0[t\sqrt{x^2 - t^2}](xt)^{\frac{1}{2}}\, dt, \tag{1}$$

then g is also square integrable and

$$f(x) = \int_x^\infty g(t) J_0[x\sqrt{t^2 - x^2}](xt)^{\frac{1}{2}}\, dt. \tag{2}$$

Here $J_0(x)$ is the Bessel function of the first kind, and the integral in (2) converges in L^2. The above relations follow from a theorem of de Branges [1, Theorem 2]. If g is related to f as in (1), we shall call g the Sonine transform (of order zero) of f. This transform and its inverse given by (2) are to some extent similar to the Sonine operators introduced by Sneddon [5; 21] in connection with the solution of dual integral equations occurring in diffraction theory.

If f is in the Lebesgue class L^p, g is well defined. The object of this paper is to extend the inversion to such functions which may not belong to $L^2(0, \infty)$.

1. Definitions and statement of results. For a measurable function f, we define

$$Sf = \int_0^x f(t) J_0[2\sqrt{t(x-t)}]\, dt \tag{3}$$

and

$$S^*f = \int_x^\infty f(t) J_0[2\sqrt{x(t-x)}]\, dt, \tag{4}$$

whenever the integrals on the right converge in some sense. It follows from (1) and (2) by simple change of variables that if f is in $L^2(0, \infty)$, then $S^*Sf = f$, a.e. It is this form of the Sonine transform that we discuss here. S is not a Watson transform, but it turns out that it can be treated in a similar manner. Our results are summed up in the following theorems.

THEOREM 1. *Let $1/p + 1/q = 1$ and let $1 \le p \le 2$. If f is in $L^p(0, \infty)$ then Sf and S^*f exist almost everywhere and are in $L^q(0, \infty)$. Also, if f and g are in $L^p(0, \infty)$, then*

$$\int_0^\infty g(Sf)\, dx = \int_0^\infty f(S^*g)\, dx. \tag{5}$$

Received August 22, 1968.