A CONDITION FOR PEAK POINTS

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For a compact set K in the plane we denote, as usual, by $R(K)$, the uniform closure on K of the rational functions with poles off K, and by $A(K)$, the algebra of those functions that are continuous on K and holomorphic on the interior of K. We let $E(K)$ denote the set of boundary points of K that do not lie on the boundary of any component of the complement of K. Recently T. A. McCullough, [5], has considered compact sets K such that

1) ∂K has finitely many components
2) $E(K)$ is countable
3) Every point of $E(K)$ is a peak point for $R(K)$.

He proves, using the methods of functional analysis, that, among other things, $A(K) = R(K)$ for such sets. We will show that 1) and 2) together imply 3).

(Since this paper was written, T. A. McCullough has given a simple direct proof that 1) and 2) imply 3); see the addendum to his paper, *Rational approximation on certain plane sets*, Pacific J. Math., vol. 29, No. 3, (1969). His method can be used to simplify the second step of the theorem of this paper.) We show more:

THEOREM. *Let K be a compact set in the plane, let K_1 be a component of K, and let L be a component of ∂K_1 such that L contains more than one point and $E(K_1) \cap L$ is countable; then every point of L is a peak point for $R(K)$.*

First we need a lemma to show that it is enough to consider the case where K is connected.

LEMMA. *Let K be a compact set in the plane and let K_1 be a component of K. Take $x_0 \in \partial K_1$, if x_0 is a peak point for $R(K_1)$. Then x_0 is a peak point for $R(K)$.*

Proof. By a theorem of Bishop [2], it is enough to show that if m is representing measure for x_0, i.e., a positive Borel measure on K such that $f(x_0) = \int f\, dm$ for all $f \in R(K)$, then $m = \delta$, the point mass at x_0. Let m be such a measure, and let K_2 be a compact subset of K such that $K_1 \cap K_2 = \phi$. Since K_1 is a component there exist compact sets $K_1' \supseteq K_1$, $K_2' \supseteq K_2$ such that $K = K_1' \cup K_2'$ and $K_1' \cap K_2' = \phi$. By Runge's Theorem there exist rational functions f_n with poles off K such that $f_n \to 0$ on K_1' and $f_n \to 1$ on K_2', the convergence being uniform on K. So we have $f_n(x_0) = \int f_n\, dm$, now $f_n(x_0) \to 0$, $\int f_n\, dm \to m(K_2')$ and hence $m(K_2) = 0$. It follows that m is carried on K_1. Now suppose f is rational with poles off K_1, since every component of $C - K_1$

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