CONVERGENCE RADIUS OF REGULARLY MONOTONIC FUNCTIONS

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1. Introduction. This paper does not assume familiarity with the literature of regularly monotonic functions [2].

Let ϵ be a function defined on $\{0, 1, 2, \cdots\}$ with values in $\{-1, 1\}$. An infinitely-often differentiable function f on an open interval I will be called ϵ-monotonic in I if for $n = 0, 1, 2, \cdots$ and all $x \in I$

$$\epsilon(n)f^{(n)}(x) \geq 0.$$

A function which is ϵ-monotonic for some ϵ is called regularly monotonic.

We define $\nu_\epsilon(n)$ for $n = 0, 1, 2, \cdots$ as follows: if there exists a least positive integer m with $\epsilon(n)\epsilon(n+1) = -\epsilon(n+m)\epsilon(n+m+1)$, then $\nu_\epsilon(n) = m$, otherwise $\nu_\epsilon(n) = \infty$. Let ϵ belong to A or B according as ν_ϵ assumes the value ∞ or not. The set $\{0, 1, 2, \cdots\}$ may be decomposed into a succession of disjoint blocks [1] by letting every n with $\nu_\epsilon(n) = 1$ be the last element of a block.

In case $\epsilon \in A$, let n be minimal such that $\nu_\epsilon(n) = \infty$. In other words, let n be the first element of the last block. If we set $\sigma = \epsilon(n)\epsilon(n+1)$, then for any ϵ-monotonic f on I, $|f^{(n)}(\sigma x)|$ is absolutely monotonic (all derivatives are non-negative) on $\{x : \sigma x \in I\}$. By using [3; Theorem 3a] one can conclude that f can be continued analytically into the open disk with centre an endpoint of I (the left one if $\sigma = 1$, the right one if $\sigma = -1$) and with radius the length of I. If the latter is infinite, the disk becomes a plane or half-plane.

In the case $\epsilon \in B$, to which we restrict our attention from here on, there are infinitely many blocks each of finite length. It can be shown that an ϵ-monotonic f must either be a polynomial or else have a finite interval as domain, so that we may choose $I = (-1, 1)$ without loss of generality.

From now on we write ν instead of ν_ϵ. Let $\lambda(k)$ be the number of elements in the k-th block. Thus $\lambda(1) = \nu(0)$, and, in general, $\lambda(r+1) = \nu(\Lambda(r))$ where

$$\Lambda(r) = \sum_{\mu=1}^{r} \lambda(\mu).$$

In this paper we shall often employ the conventions that the empty sum has the value 0 and the empty product the value 1. Here, for example, $\Lambda(0) = 0$, and in 2.5 $\Pi(0) = 1$.

Our goal is

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