LOCALLY CONVEX TOPOLOGIES ON FUNCTION SPACES

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1. Introduction. We consider a vector space $G(X)$ of scalar-valued functions on a set X. We show in Theorem 1 that if G equipped with a topology $\mathfrak{T}$ is a locally convex (topological vector) space, then $\mathfrak{T}$ can be described in terms of convergence on filters [2; 287, Def. 1.2]. Our interest in this result centers on the fact that the filters are on the domain space X. Elsewhere we will show that, using tensor products, our methods and results carry over to operator spaces, and give access to approximation theorems for several classes of linear operators.

Convergence on filters generalizes uniform convergence. It appears to be a versatile tool in connection with spaces of functions which can be viewed as having two or more domains. In particular, convergence on filters is a flexible adjunct to uniform convergence in the duality theory of locally convex spaces. These remarks can be illustrated by our proof of Theorem 1, which is our basic result. We first express the topology $\mathfrak{T}$ on G as uniform convergence on the equicontinuous subsets of the dual space G'. This is automatically a topology of convergence on filters in G^*, the algebraic dual. The crucial step then is a coarsening of these filters, without change of topology, so that each new filter $\mathfrak{F}$ generates a pre-image filter $e^{-1}(\mathfrak{F})$ in X. Here e is the evaluation (natural) map of X into G^*. The filters $e^{-1}(\mathfrak{F})$ suffice to establish our theorem. We observe below that $e(X)$ may not meet G'; from this it is clear that a result like Theorem 1 cannot be obtained simply by pulling equicontinuous subsets of G' back to X.

The basic Theorem 1 can be applied to obtain descriptions of many topologies on $G(X)$ defined in terms of the duality of G and G'. We present results of this kind for the weak topology $\sigma(G, G')$ and the Mackey topology $\tau(G, G')$. We turn now to preparatory material.

2. Preliminaries. It is convenient to develop many of our results under the assumption that X is a vector space and $G(X)$ a space of linear forms on X; that is, G is a subspace of the algebraic dual X^* of X. In the absence of these conditions we may take a new view of G as a space of linear forms defined on the linear span of $e(X)$ in the algebraic dual G^* of G. By this device we avoid cumbersome circumlocutions in dealing with an essentially linear situation; see [1; 649, Def. 5.3]. We also present matters in a form closest to, and most useful for, our later work on operator spaces.

We take from [2] our convergence machinery in a form adapted to present needs. We will refer interchangeably to a filter of subsets of X as a filter *on* X

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