GEOMETRIC EMBEDDING INVARIANTS OF SIMPLE CLOSED CURVES IN THREE—SPACE

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1. Introduction. This paper is devoted to the study of the relationships between three geometric embedding invariants of a simple closed curve in three-dimensional Euclidean space: the penetration index, the enveloping genus, and the local enveloping genus. See §2 for definitions. Attention will be focused on the simplest nontrivial case, in which the simple closed curve is assumed to be locally tame modulo a single point.

The work presented here was motivated by some questions raised by Ball [2; 37–38], which are answered in §4 and §5. Also, see Question 19 of [8], which is partially answered by §5. For the statements of the results of this paper, see §3.

2. Definitions and notations. The setting for all of our results is Euclidean three-space, which we denote by E^3. Throughout the entire paper, K shall denote a simple closed curve in E^3 which is locally tame modulo a point q. Examples of such simple closed curves are well known. For instance, see [1], [9], [10] and [14], in all of which the examples are described as arcs rather than simple closed curves. By [15, Theorem 0], there is no loss of generality in supposing that K is locally polygonal modulo q, and we shall do this. For definitions of *locally tame, locally polygonal*, etc., see [3].

(2.1) Definition. Let A be a 1-dimensional subset of E^3 and $p \in A$. The *penetration index* of A at p, denoted by $P(A, p)$, is defined to be the smallest cardinal number ν such that there are arbitrarily small 2-spheres whose interiors contain p and which intersect A in no more than ν points. This definition is due to Alford and Ball [1] and is an extension of Harrold's property $\mathcal{P}$ of [11]. We define the *penetration index* of A, denoted by $P(A)$, to be the least upper bound of the set of cardinal numbers $\{P(A, p) \mid p \in A\}$. Notice that in our case, $P(K) = P(K, q)$.

(2.2) Definition. Let A be a 1-dimensional continuum in E^3. Then there exists a sequence $\{M_i\}_{i=1}^{\infty}$ of compact connected 3-manifolds with connected boundaries such that if $i = 1, 2, \cdots$, then $M_{i+1} \subset \operatorname{Int} M_i$, and $A = \bigcap_{i=1}^{\infty} M_i$. The sequence $\{M_i\}_{i=1}^{\infty}$ is said to be a *defining sequence* for A. The *enveloping genus* of A, denoted by EG (A), is defined to be the smallest integer m such that there exists a defining sequence $\{M_i\}_{i=1}^{\infty}$ for A such that if $i = 1, 2, \cdots$, then

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