ON THE MEAN PARITY OF ARITHMETICAL FUNCTIONS

By Eckford Cohen

To Leonard Carlitz on his sixtieth birthday.

1. Introduction. Let $f(n)$ denote a complex-valued function on the positive integers n, and let b and m represent integers with b assumed >0. In this paper, we study the regularity of the mean value of certain functions f when averaged over the set $P_{m,b}$ of integers n of $\equiv m \pmod b$.

The (ordinary) mean value of f is, by definition, the limit

$$M(f) = \lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} f(n), \tag{1.1}$$

provided this limit is known to exist. If in (1.1), n is restricted to the arithmetical progression $P_{m,b}$, then the corresponding limit (see §2), to be denoted $M_{m,b}(f)$, is evidently *periodic* as a function of $m \pmod b$. In §2 we show that, for certain functions f, $M_{m,b}(f)$ is not only periodic but is in fact *even* $\pmod b$. A function $F(m, b)$ is said to be even as a function of $m \pmod b$ if $F(m, b) = F((m, b), b)$ for all m, where (m, b) denotes, as usual, the greatest common divisor of m and b.

Our attention is mainly directed to two special classes of functions f, one of "divisor" type and the other of "totient" type. Quite simple formulas for the mean values of these functions are determined in §3. It is of interest that for such functions f, the mean value $M_{m,b}(f)$ can be identified with certain subclasses of the even functions $\pmod b$ introduced by P. J. McCarthy [7]. For a clarification of the term "mean parity" in an amplified form, the reader is referred to the definition of §3.

In §§4 and 5 we are concerned with trigonometric expansions related to the special functions discussed in §3. In particular, the parity property $\pmod b$ is shown (§4) to lead to finite Fourier representations for the mean $M_{m,b}(f)$, and from these finite representations of the mean value, infinite series expansions of the functions f are easily deduced (§5). In this discussion a fundamental role is performed by the author's generalization $c_k(n, r)$ of Ramanujan's trigonometric sum [2]. Here k is a positive integer and r a positive integral variable, $b = r^k$.

Finally, the reader's attention is drawn to Theorem 5 (§5), which reveals a surprising relation between the functions under discussion and their mean values. It will be noted that this result implies that the functions of this paper

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