MATRICES OF SCHUR FUNCTIONS

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1. Statements. In a recent announcement [1], J. E. de Pillis stated the following result:

*Let H be an $mn \times mn$ positive semi-definite hermitian matrix, and partition H into m^2 $n \times n$ matrices H_{st}, $s, t = 1, \cdots, m$. Let $1 \leq q \leq n$ and let e_{st} denote the q-*th *elementary symmetric function of the eigenvalues of the matrix H_{st},* i.e., *$e_{st} = E_q(H_{st})$. Then the $m \times n$ matrix $E = (e_{st})$ is positive semi-definite hermitian also.*

In the present paper we prove a substantial generalization of this theorem as a consequence of an elementary lemma on traces of submatrices of hermitian matrices.

In order to state our result we introduce a general class of polynomial functions known as Schur functions. Thus let G denote a subgroup of the symmetric group of degree p, S_p, $1 \leq p \leq n$, and let χ be a non-zero character of degree one on G. Define an equivalence relation "$\sim$" on the set $\Gamma_{p,n}$ of all n^p sequences $\omega = (\omega_1, \cdots, \omega_p)$, $1 \leq \omega_i \leq n$, $i = 1, \cdots, p$, as follows: two sequences α and β are equivalent, i.e., $\alpha \sim \beta$, if and only if there exists $\sigma \in G$ such that

$$\alpha^\sigma = (\alpha_{\sigma(1)}, \cdots, \alpha_{\sigma(p)}) = \beta.$$

Let Δ_n denote the system of distinct representatives in $\Gamma_{p,n}$ for "$\sim$" in which each sequence α in Δ_n is lowest in lexicographic order in the equivalence class in which it lies. Define a subset $\bar{\Delta}_n$ of Δ_n as follows: $\bar{\Delta}_n$ is the set of all sequences $\alpha \in \Delta_n$ for which $\chi \equiv 1$ on the stabilizer G_α in G. Here $G_\alpha = \langle \sigma \mid \sigma \in G$ and $\alpha^\sigma = \alpha \rangle$. Let $\nu(\alpha)$ denote the order of G_α. The Schur function associated with G and χ is the polynomial

$$f_{G,\chi}(\gamma_1, \cdots, \gamma_n) = \sum_{\omega \in \bar{\Delta}_n} \prod_{t=1}^{n} \gamma_t^{m_t(\omega)}, \tag{1}$$

in which $m_t(\omega)$ is the number of times the integer t occurs in ω. If X is an $n \times n$ matrix, we define

$$f_{G,\chi}(X)$$

to be $f_{G,\chi}(\gamma_1, \cdots, \gamma_n)$, where $\gamma_1, \cdots, \gamma_n$ are the eigenvalues of X.

The first main result is the following:

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