C^1 FUNCTIONS ON A COMPLEX ANALYTIC VARIETY

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Introduction. Holomorphic functions on a complex analytic variety are defined via restriction of holomorphic functions in the ambient space. One may similarly define C^∞ or C^m ($m = 1, 2, \cdots$) functions on a complex analytic variety. For example, suppose that V is a complex analytic subvariety of an open subset U in $\mathbf{C}^n$. Then f is a C^∞ function on V if there exists a C^∞ function g on a neighborhood of V in U such that $g/V = f \cdots$ [2].

Malgrange [8] and Spallek [9], [10] have studied these functions and their relation to complex analytic functions. Malgrange proved that if f is a weak holomorphic function (i.e. f is continuous and holomorphic at the regular points) and f is C^∞, then f is holomorphic. Spallek showed the following: For any point p on a complex analytic variety V, there exists an integer M such that if f is weak holomorphic in a neighborhood of p and f is C^m for $m \geq M$, then f is holomorphic near p. Of course, precisely what degree of differentiability is required to insure that f be holomorphic depends on the singularity of V at p. In §4, we will give an example of a weak holomorphic, C^1 function which is not holomorphic. This then, will be an example of a singularity where the critical integer M is > 1.

Let $\mathfrak{D}^1$ denote the sheaf of germs of real-valued C^1 functions on a complex analytic variety V. This sheaf may be used to define a tangent space to the complex analytic variety V at a point $p \in V$. This tangent space will depend only on the germ of V at p and we will denote it by $T(V, \mathfrak{D}_p^1)$. We will study the relation between this tangent space and $T(V, \mathcal{O}_p)$, the tangent space derived from the sheaf of germs of holomorphic functions $\cdots$ [4].

We will show that $T(V, \mathfrak{D}_1^p)$ is isomorphic to a complex vector subspace of $T(V, \mathcal{O}_p)$. For all $p \in V$ we obtain the inequalities:

$$\dim_p V \leq \dim_C T(V, \mathfrak{D}_p^1) \leq \dim_C T(V, \mathcal{O}_p).$$

p is a regular point of V, if and only if,

$$\dim_p V = \dim_C T(V, \mathfrak{D}_p^1) = \dim_C T(V, \mathcal{O}_p).$$

If p is a singular point of V, then:

$$\dim_p V < \dim_C T(V, \mathfrak{D}_p^1) \leq \dim_C T(V, \mathcal{O}_p).$$

In §3.11 we give an example of a variety V, and a point $p \in V$, such that $\dim_C T(V, \mathfrak{D}_p^1) < \dim_C T(V, \mathcal{O}_p)$. This will, again, be an example where there is a deviation between the C^1 structure and the complex analytic structure.

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