THE AUTOMORPHISM GROUP OF A LOCALLY COMPACT GROUP

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Introduction. Let G be a locally compact topological group, and let $A(G)$ denote the group of all topological group automorphisms of G. In general $A(G)$ with the compact-open topology is not a topological group and some finer topology has to be adapted. If K is a compact subset of G and V is a neighborhood of 1 in G, we denote by $N(K, V)$ the set of all elements α of $A(G)$ such that $\alpha(x)x^{-1}$ and $\alpha^{-1}(x)x^{-1}$ belong to V for every x in K. Then the sets $N(K, V)$ constitute a fundamental system of neighborhoods of the identity which gives the usual topological group structure of $A(G)$. It is well known that when G is either a compact group or a Lie group, the topology of $A(G)$ coincides with the compact-open topology. In this paper we shall study the following question: For which groups is this topology simply the compact-open topology? For sake of briefness, we say that G has property (P) if the usual topology of $A(G)$ is the compact-open topology. Our main result is that G has property (P) provided G/G_0 has property (P) where G_0 is the identity component of G.

1. Limit of net of elements of $A(G)$. It is clear that G has property (P) if and only if for any net $\{\alpha_\lambda\}$ of elements of $A(G)$ converging to the identity automorphism 1_G with respect to the compact-open topology, $\{\alpha_\lambda^{-1}\}$ converges to 1_G with respect to the compact-open topology. In this section, we study limit of net of elements of $A(G)$ with respect to the compact-open topology.

Lemma 1.1. *Let $\{\alpha_\lambda\}$ be a net of elements of $A(G)$ converging to 1_G with respect to the compact-open topology, and N be a compact normal subgroup of an open subgroup H of G such that H/N is a Lie group. Then $\alpha_\lambda(N) \subset N$ holds for large λ.*

Proof. Let V be a neighborhood of 1 in H such that VN/N has no small subgroups in H/N. Since $\{\alpha_\lambda\}$ converges to 1_G with respect to the compact-open topology, $\alpha_\lambda(N) \subset VN$ for $\lambda > \lambda_0$. Hence $\alpha_\lambda(N) \subset N$ for $\lambda > \lambda_0$.

Lemma 1.2. *Let $\{\alpha_\lambda\}$ be a net of elements of $A(G)$ converging to 1_G with respect to the compact-open topology and suppose the set $\{\alpha_\lambda^{-1}(x_0) : \lambda > \lambda_0\}$ has compact closure in G. Then the net $\{\alpha_\lambda^{-1}(x_0)\}$ converges to x_0 in G.*

Proof. Let K be a compact subset containing the set $\{\alpha_\lambda^{-1}(x_0) : \lambda > \lambda_0\}$. Given any symmetric neighborhood V of 1, $\alpha_\lambda(k)k^{-1} \in V$ holds for all $k \in K$ and $\lambda > \lambda_1$. Hence in particular, $x_0\alpha_\lambda^{-1}(x_0^{-1}) = \alpha_\lambda(\alpha_\lambda^{-1}(x_0))\alpha_\lambda^{-1}(x_0^{-1}) \in V$, for all $\lambda > \lambda_1$. It follows that $\alpha_\lambda^{-1}(x_0) \in Vx_0$, for $\lambda > \lambda_1$, and we get $\lim_\lambda \alpha_\lambda^{-1}(x_0) = x_0$.

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