ON NON-MEASURABLE SETS

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In this paper we discuss the existence and structure of non-measurable sets in a Romanovski space (a second countable, locally compact metric space X on which is defined a complete regular measure μ, positive on non-void open sets, for which there is a base $\mathfrak{A}$ of the topology of X satisfying ten axioms given by P. Romanovski in 1941 (see, e.g., [6; §1.2]). To this end, we introduce a concept of "non-measurable kernel" in the space X and show that the non-measurable behavior of a set $E \subseteq X$ is, in a certain sense, totally governed by that of a non-measurable kernel contained in it. For all Romanovski spaces X it is shown that every set $E \subseteq X$ of positive outer measure is either a non-measurable kernel or can be written as the union of two disjoint non-measurable kernels each of which has outer measure equal to that of E. It should be recalled that every finite-dimensional Euclidean space, E^n, with Lebesgue measure and proper choice of $\mathfrak{A}$, is an example of a Romanovski space, so that our results are applicable to all such spaces. Throughout this paper, the notation, definitions and conventions of [6] and [7] will be adopted.

I. Non-measurable kernels.

1. Definition. We call a non-measurable set $E \subseteq X$ a *non-measurable kernel* if $H \subseteq E$ and H measurable implies H is of measure zero.

Alternatively, a set E having positive outer measure is a non-measurable kernel if and only if E^c intersects every measurable set of positive measure.

2. Theorem. *Let $E \subseteq X$ be non-measurable. Then there is a non-measurable kernel $E^* \subseteq E$ such that:*

(1) *for every set $H \subseteq X$, $\mu^*(H \cap E) = \mu^*(H \cap E^*) + \mu^*([E - E^*] \cap H)$;*

(2) *$E - E^*$ is measurable.*

Specifically, E^* may be chosen to be

$$\{x \in E: \bar{\Phi}^*(E^c, x) > 0\} = \{x \in E: \Phi^*(E^c, x) = 1\} \cup Z,$$

where $\mu^*(Z) = 0$ (for the definition of $\bar{\Phi}^*$ and Φ^*, see [7, II].

Proof. Let E be non-measurable, $E_1 = E$ and $E_2 = E^c$,

$$E_i(E_j) = \{x \in E_i : \bar{\Phi}^*(E_j, x) > 0\},$$

$i \neq j$, $i = 1, 2$. Then, by [7, Theorems 1.2 and 2.6], $\mu^*(E_1(E_2)) = \mu^*(E_2(E_1)) > 0$. From ([7], Theorem 2.9], $\Phi^*(E_j, x) = 1$ for almost all $x \in E_i(E_j)$. Be-

Received September 7, 1967. The research for this paper was supported by NSF grant no. GP-6815.