G_δ SECTIONS AND COMPACT-COVERING MAPS

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1. Introduction. A continuous map $f : E \to F$ is called *compact-covering* if every compact subset of F is the image of some compact subset of E. Bourbaki [2; 45] proved that f is always compact-covering if f is open, E complete metric, and F Hausdorff. In [4; Corollary 1.2] it is shown that the completeness of E can be weakened to:

(*) For some metric on E, each $f^{-1}(y)$ is complete.

Example 4.1 in [4] shows, moreover, that (*) cannot be replaced by the weaker requirement that each $f^{-1}(y)$ is completely metrizable. The space E of that example was, however, obtained by non-constructive methods, and was thus—presumably—not Borel. The purpose of this note is to construct an example in which E is actually σ-compact; since E cannot be complete, that is as far as one can hope to go in this direction.

Our example will be constructed with the aid of a theorem on special G_δ sections for continuous maps between compact metric spaces, which may have some independent interest. (That such maps always possess G_δ sections was proved by Bourbaki [2; 144, Example 9a].

In addition to the works cited above, compact-covering maps have been studied in [1], [5], and [6].

2. Sections. A *section* for a map $h: P \to Y$ is any subset $S \subset P$ which intersects every non-empty $h^{-1}(y)$ in exactly one point.

Throughout this section, X and Y will denote compact metric spaces, with Y uncountable, and π a continuous map from X onto Y.

LEMMA 2.1. *There exists a closed $B \subset X$ such that every section for $\pi \mid B$ intersects every closed $A \subset X$ for which $\pi(A) = Y$.*

Proof. Since Y is uncountable, it has a subset K homeomorphic to the Cantor set. Let $\mathfrak{F}(X)$ denote the space of all non-empty, closed subsets of X, topologized with the Hausdorff metric, and let

$$\mathfrak{I} = \{A \in \mathfrak{F}(X) : \pi(A) = Y\}.$$

Then $\mathfrak{I}$ is closed in $\mathfrak{F}(X)$; since X is compact metric, so is $\mathfrak{F}(X)$ [3; §38, I, 1], and hence so is $\mathfrak{I}$. Also $X \in \mathfrak{I}$, so $\mathfrak{I}$ is non-empty. Hence there exists a continuous g from K onto $\mathfrak{I}$. Let

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